THE MOTION AND THE DISTRIBUTION OF THE STARS

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THE MOTION AND THE DISTRIBUTION OF THE STARS

BY

CARL WILHELM LUDVIG CHARLIER

Professor of Astronomy and Director of the Observatory, Lund, Sweden

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Dedicated to
Professor Armin O. Leuschner
Director of the Students' Observatory
of the University of California
Berkeley, California

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PREFACE

At the kind invitation of the President of the University of California, Dr. W. W. Campbell, these lectures were delivered in Berkeley from the 12th of May to the 1st of August, 1924. They were intended to give, in condensed form, the results of my researches on stellar statistics and especially on the motion and the distribution of the stars.

In order that the mathematical part of the lectures might be more easily comprehended, it was necessary to give a short summary of the elements of mathematical statistics. This summary is *not* included here, and the reader is referred to my *Grundzüge der mathematischen Statistik* (1920).[1]

It was my original intention to publish these lectures as a textbook on Stellar Statistics, in direct continuation of my *Introduction to Stellar Statistics* (1921).[2] I could not, however, decline the flattering offer of the University of California to publish my Berkeley lectures in the *Memoirs* of that university, an invitation for which I express my sincere thanks.

In order that they may more easily follow these lectures, readers are recommended to familiarize themselves with the main contents of the two small books mentioned above. In the *Grundzüge* the principal theorems in mathematical statistics are developed and exemplified, while in the *Introduction* the elementary conceptions of stellar statistics are explained.

To Professor W. F. Meyer, one of my auditors in Berkeley, who has with great care and at the cost of valuable time read the first two proofs of these lectures, I express my heartfelt thanks.

It is a great pleasure to me to have the opportunity of dedicating these memoirs to Professor A. O. Leuschner, Director of the Students' Observatory in Berkeley. It was due to his repeated suggestions and invitations that I made, in 1924, my trip to the United States, of which I have so many happy and valued memories, in connection not only with my visits to the admirable observatories of that country, but also with my pleasant association with the astronomical students of the universities of California.

The Observatory of Lund, the 15th of December, 1925.

C. V. L. Charlier.

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[1] C. V. L. Charlier: *Vorlesungen über die Grundzüge der mathematischen Statistik.* Verlag Scientia, Lund.
[2] C. V. L. Charlier: *Introduction to Stellar Statistics.* Scientia, Lund.

PART I

THE MOTION OF THE STARS

CHAPTER I

THE APPARENT MOTIONS OF THE STARS

A. Motion in the Line of Sight

1. The apparent motion of the stars consists of a secular motion, which always takes place in the same direction, and a periodical motion, which depends upon the periodic motion of the observer (on the earth) or of the star. For the present, the periodical motion will be left out of consideration. The secular motion consists in part of an apparent motion perpendicular to the line of sight—the so-called proper motion of the stars—and in part of a motion along the line of sight, the *radial velocity* of the stars. The latter, which is measured, according to the principle of Doppler, through the displacement of the lines in the spectrum, takes place in a single dimension. It is for this reason easier to discuss, and will therefore be considered first.

We shall choose a limited portion of the sky (the square GC_2, including Hercules and the Swan), and shall study the radial velocities in this region. We shall limit our research to stars brighter than the sixth magnitude, the only stars for which proper motions and radial velocities have been closely examined. In the square GC_2 there are in all 145 stars brighter than the sixth magnitude. Of these the proper motions are known for 139 stars, the radial velocities for 46. Later observations, especially at the Lick Observatory, have made available the radial velocities of a greater number of stars. This additional material will be discussed later.

The values obtained for the radial velocities of these stars—most of them determined by Campbell—are the following. All velocities are expressed in Siriometers per stellar year. The plus sign indicates that the star is receding from us, the minus sign that it is approaching us. One star was accidentally omitted from this table, so that the number of stars treated is 45, instead of 46.

Radial velocities of the stars in the square GC_2

−5.70	−1.73	−7.39	−0.74
−5.47	−5.66	−5.49	−5.85
−4.31	−4.22	+1.27	−8.19
−2.20	−4.98	−0.04	−0.84
−2.91	+0.84	−0.84	−1.46
−5.49	−7.35	−4.64	−0.82
+4.31	−1.90	−2.11	−3.42
−1.65	−6.40	−4.29	−3.23
−2.72	−6.12	−0.59	−5.40
+1.48	−3.80	−1.08	
−5.64	+0.84	−3.80	
−4.64	−5.11	−6.76	

Only five of these stars are receding from us, the other 40 are approaching more or less speedily. The greatest velocity toward us is 8.19 Sir/st (appr. = 40 km/sec), from us 4.31 Sir/st (= 20 km/sec).

The number is too small for a thorough statistical research, but rather plausible values may be obtained for the lowest statistical characteristics, the *mean* and the *dispersion*. We find that the mean velocity amounts to −3.25 Sir/st and the dispersion to 2.69 Sir/st (the mean error of the mean amounts to ±0.40, and of the dispersion to ±0.28), so that these 45 stars have an average velocity toward us of 3.25 Sir/st (= 15 km/sec), the single values being dispersed around this mean value with a standard deviation or dispersion of 2.69 Sir/st.

W	Number
−8	1
−7	3
−6	6
−5	8
−4	5
−3	4
−2	5
−1	7
0	1
+1	4
+2	0
+3	0
+4	1

The distribution of the observed velocities may be seen in the small table here given. Little more, of course, could be expected from this small material. We notice only that, in full compliance with the mathematical character of the dispersion, about two-thirds (31) of the whole number of stars lie between the limits obtained through adding and subtracting the dispersion from the mean, i. e., between the limits −5.94 and −0.56.

2. To gain a more complete conception of the radial velocities of the stars we must go to other parts of the sky. In table 1 are the values I have obtained for the mean values of the radial velocities of the stars and of their dispersion for each of the 48 galactic squares.

This table embraces all stars (m <6.00) for which the radial velocities were known to the autumn of 1922. The total number of these stars is 2177. In a statistical discussion of these observed values it is, however, necessary to exclude those stars that possess a comparatively great velocity, either in the radial component W or in the proper motions u and v. For this reason I have excluded all stars having a velocity component deviating more than three times the dispersion from the mean value of the component in each square. In this manner 191 stars were eliminated so that 1986 stars remain to be discussed.

W	Number of squares
−4	1
−3	4
−2	9
−1	5
0	8
+1	3
+2	9
+3	3
+4	5
+5	1

The first column gives the galactic square, the second, the number of stars,the third, the mean value of the radial velocity, the fourth column, the dispersion.

The mean value of W for all squares being +0.173 Sir/st, we conclude that the stars are moving from the sun. For a discussion of this *expansion* of the stars, a more detailed table is needed than the one here given. I shall therefore return to this important question in the next chapter.

The distribution of the mean values of the radial velocities in the different squares may be seen from the accompanying table. The values of W are rather symmetrically distributed about the value $W = 0$, the small values of W occurring most frequently.

The most interesting conclusion, however, is connected with the distribution of these values in the sky, which is described rather clearly by means of diagram 1 in the projection of Flamsteed.

TABLE 1

Mean value and dispersion of the radial velocity of the stars in different squares

	N	$\overline{W}$	σ		N	$\overline{W}$	σ
GA_1	26	−1.71	3.38	GD_1	53	−3.47	3.11
GA_2	35	−0.74	3.09	GD_2	53	−2.24	3.24
				GD_3	53	−1.59	2.77
GB_1	50	−3.33	4.03	GD_4	67	−0.48	3.47
GB_2	53	−2.78	3.52	GD_5	60	+3.14	3.24
GB_3	48	−1.37	2.81	GD_6	82	+4.54	2.97
GB_4	46	−0.44	4.10	GD_7	46	+4.94	3.56
GB_5	33	+1.10	3.56	GD_8	49	+4.08	2.14
GB_6	38	+1.84	4.18	GD_9	70	+2.80	3.10
GB_7	21	+0.88	3.59	GD_{10}	33	+0.73	2.96
GB_8	17	−1.30	2.69	GD_{11}	47	−0.39	4.24
GB_9	19	−1.15	3.67	GD_{12}	40	−1.81	3.76
GB_{10}	42	−3.97	4.19				
				GE_1	24	−0.88	3.49
GC_1	55	−3.99	3.20	GE_2	34	−1.30	2.88
GC_2	74	−3.35	2.77	GE_3	37	−0.23	2.87
GC_3	56	−2.68	3.74	GE_4	40	+1.53	3.54
GC_4	50	−1.22	3.30	GE_5	35	+2.46	4.00
GC_5	41	+0.84	3.61	GE_6	27	+2.48	3.14
GC_6	52	+2.21	4.30	GE_7	33	+3.23	3.08
GC_7	27	+3.72	2.94	GE_8	29	+2.05	2.95
GC_8	25	+4.08	3.42	GE_9	27	+1.97	3.70
GC_9	37	+1.68	2.76	GE_{10}	33	+0.72	3.58
GC_{10}	45	+0.17	3.10				
GC_{11}	47	−2.24	3.07	GF_1	19	+1.11	3.74
GC_{12}	33	−2.50	4.15	GF_2	25	+2.11	2.89
				Σ	1986	+8.287	
				Mean	41	+0.1726	3 366

From this diagram it is evident that the different values of W are not distributed at random in the sky, but, on the contrary, their distribution possesses a very pronounced and prominent characteristic. The positive values—belonging to stars which are receding from us—are with conspicuous regularity grouped round the square GD_7 (constellation Lepus, the Hare), whereas the stars which are approaching our solar system are to be found in the environment of the square GC_1*

* It is to be noted that the squares GC_1 and GC_{12} are in close proximity and that the squares together cover the whole sphere.

(constellation Hercules). If the values in the diagram were projected upon a sphere, we should find that the squares with a very small or evanescent radial velocity form, approximately, a great circle on the sphere, being at the same distance from the two squares with the maximum and minimum velocities. We therefore conclude from the diagram that the stars about Lepus are receding from us, while the stars in the opposite part of the sky are approaching our system (we may say that the Hare is running away from the stars, which are being hunted by Hercules!).

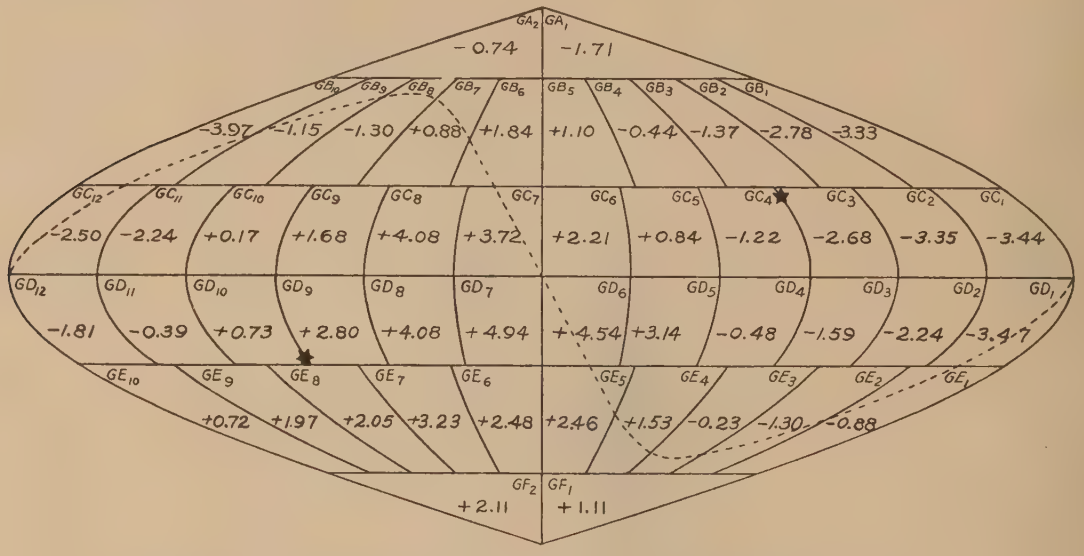

Diagram 1.—Mean Value of the Radial Velocity of the Stars in Different Squares.

We may draw another conclusion of a general nature from our diagram, namely, that the stars about Lepus recede from us with a greater velocity (+4.9 Sir/st) than that with which the stars in Hercules approach us (−4.0). This is, indeed, nothing but another expression of the thesis to which we have already alluded, and which will be discussed in more detail in the next chapter; namely, that the stars *on an average* are receding from us. The conclusion here, however, is more decisive. From the clearly marked regularity in the radial velocities about the maximum and minimum values, we may directly infer that it is not a question of mere chance.

3. Regarding the dispersion in the radial velocities it is more difficult to arrive at similar general conclusions. As already mentioned, the mean value of the dispersion amounts to 3.366. Its greatest value is 4.30 Sir/st in the square GC_6, its lowest value 2.14 Sir/st in the square GD_8. In denoting the values on a diagram, a certain regularity in the distribution of the values of σ for the different squares is suggested, but not with the same evidence as regarding the mean values.

The systematic character in the distribution is conspicuous if the means are taken of σ in two opposite squares all over the sky. We then obtain the distribution given in diagram 2.

In this diagram I have subtracted from the dispersion in each square the mean value 3.37 of the dispersion for the whole sky.

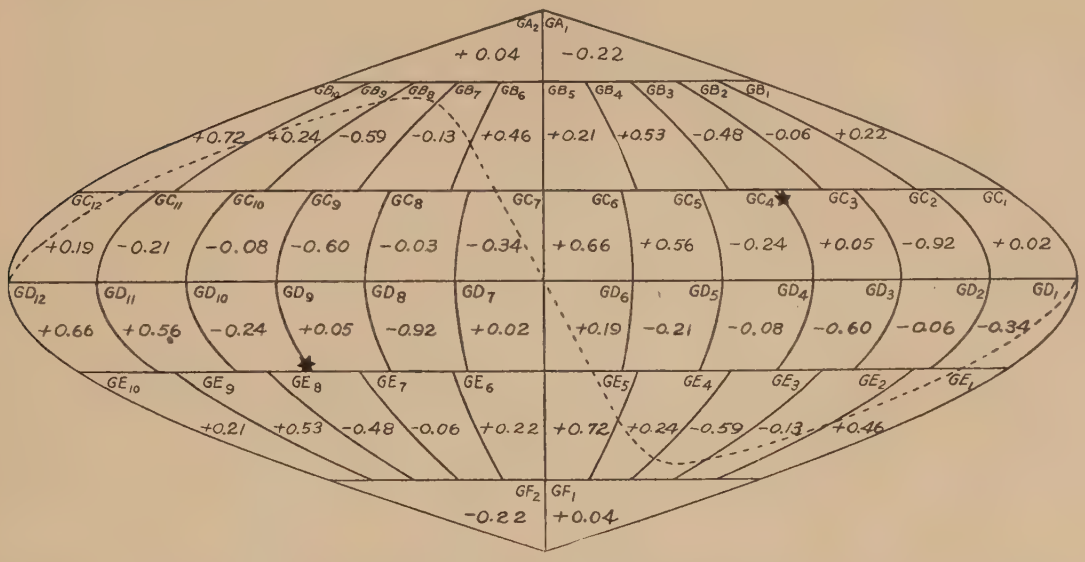

Diagram 2.—Dispersion (σ) in W (1986 stars) Diminished by the Mean Value of $\sigma=3.37$. Opposite Squares Taken Together.

It is seen from this diagram that the dispersions in the radial velocities have a pronounced maximum in the squares GC_6 and GD_{12}. At some distance from these points the dispersion is smaller than the mean value. We will find in the next chapter the explanation of this systematic variation in the dispersion of the radial velocities.

4. *Summary.*—The observations of the radial velocities of the stars have led to the following conclusions:

(1) It is suggested that the stars on an average are receding from us.

(2) The mean values of the radial velocities in the different squares show a systematic regularity, with the result that the stars about the square GC_1 (Hercules) approach us, and the stars about the square GD_7 (Lepus) recede from us.

(3) Even in the dispersion of the radial velocities a similar regularity is suggested, showing that the greatest dispersion occurs in the square GC_6 (Gemini) and GD_{12} (Sagittarius).

B. Proper Motions of the Stars

5. Let us now consider the proper motions of the stars. We shall use galactic coördinates and treat the proper motions of the stars in galactic longitude and latitude. We begin with an individual square, the same as for the radial velocities, namely, GC_2. First, I shall show how the proper motions of a group of stars (within a square) may be graphically examined.

For this purpose we suppose all stars transported to the center (O) of a square and mark out from O a line OP so that the projection of OP on the galactic latitude circle is equal to the proper motion of the star in latitude (v) and its projection on a parallel circle through O is equal to the proper motion in longitude multiplied by the cosines of the latitude. Instead of marking out the whole line OP, it suffices to set off only the point P. For every star we obtain in this manner a point in the so-called velocity diagram. To avoid the prolixity of treating every star in such a diagram individually, we divide it into quadratic fields and count the number of stars within each field. In this manner we obtain for the square GC_2 a statistical velocity diagram as shown in table 2.

TABLE 2

Statistical diagram of the proper motions of the stars in the square GC_2 ($m<6.00$)

V \ U	+7	+6	+5	+4	+3	+2	+1	0	−1	−2	−3	−4	Σ
	+0″.35	+0″.30	+0″.25	+0″.20	+0″.15	+0″.10	+0″.05	0″.00	−0″.05	−0″.10	−0″.15	−0″.20	
+3 +0″.15				1									1
+2 +0″.10			1										1
+1 +0″.05					1	2	2	3			1		9
+0 0·00				1	1	4	13	70	10		1	1	101
−1 −0″.05	1		1		1	1	4	6	3	1	1		19
−2 −0″.10	1						1	1			1		4
−3 −0″.15													
−4 −0″.20													
Σ	2	—	2	2	3	7	20	80	13	1	4	1	135

The positive X-axis (u-axis) in the diagram is laid to the left, instead of the right as usually, because in the projection of Flamsteed, which is used here for the statistical representation of the sky, the galactic longitude is increasing to the left. The class-breadth in the correlation table is taken as 0″.05. The classes are designated partly through current numbers, and partly through the middle value of each class expressed in seconds of arc per year. Thus the subclass (+2,−1)

designates that subclass for which the middle value corresponds to $u = +0.10$ and $v = -0''.05$, and generally (m, n) designates that subclass for which the middle value is $u = m\ 0''.05$, $v = n\ 0''.05$, and which extends $0''.025$ to each side of this middle value. The numbers in the diagram designate the number of stars in each subclass.

It was mentioned before that within the square GC_2 there are in all 139 stars brighter than the sixth magnitude for which the proper motions are known (according to Boss' G. C.). In the diagram only 135 stars are taken into consideration, because 4 stars with relatively large proper motions are excluded on mathematical grounds. The proper motions of these stars are the following:

EXCLUDED STARS

(185332)	,	$u = -0''.070$	,	$v = -0''.224$
(194233)	,	$u = -0.378$	,	$v = -0.247$
(200235)	,	$u = -0.493$	,	$v = -0.055$
(204233)	,	$u = +0.480$	,	$v = -0.068$

The diagram indicates there are in the square GC_2 70 stars in the subclass (0, 0), which have a proper motion in galactic longitude between the limits $-0''.025$ and $+0''.025$ and galactic latitude between the same limits. There are 10 stars in the subclass $(-1, 0)$, which have a proper motion in longitude between $-0''.025$ and $-0''.075$ and in latitude between $-0''.025$ and $+0''.025$, etc.

Let us now analyze the diagram.

The mean values of u and v are

$$M(u) = +0''.0163, \quad M(v) = -0''.0048,$$

which give the amount of the proper motion on an average for the stars considered here.

This is the first approximation. In the next approximation the dispersion in the proper motions of the stars is taken into account and finally the correlation between the proper motions in longitude and latitude.

For the dispersions in l and b we obtain, according to the methods of mathematical statistics, the values

$$\sigma_l = 0''.0757, \qquad \sigma_b = 0''.0322.$$

We further find from the diagrams of the proper motions that their distribution is by no means uniform in all directions. If that were the case the numbers in the diagram would, indeed, be (approximately) symmetrically distributed about a center determined by the mean values of u and v found above. Instead of that the distribution surface has an extended form. Roughly speaking, we may say that there is a symmetrical direction, somewhat inclined along the u-axis, which gives to the proper motion diagram approximately an elliptical form. The dispersion along the greater axis of the ellipse (more strictly parallel to the u-axis)

amounts to 0".0757 and in the perpendicular direction to 0".0322. The line of symmetry (the longer axis) makes an angle of 2°.06 with the u-axis and is hence nearly parallel to the Milky Way.

The analysis of such a diagram is performed as in mathematical statistics (analysis of a correlation surface). The problem here, indeed, concerns such a surface. Regarding the methods followed, I refer to *Meddelanden från Lunds Observatorium* (L. M.), Ser. II, Nos. 4 and 9, and to my lectures on mathematical statistics.

The foregoing description gives only incompletely the characteristic features of the proper motion diagram. There is a third approximation, and we therefore have to compute the *skewness* and the *excess*, and the different correlation coefficients of the third and the fourth order. Without further calculation we find directly from the diagram that the distribution possesses a pronounced skewness and even has an excess, and that evidently it is *positive*, as the number of stars not moving at all (or, more accurately, having a proper motion numerically smaller than 0".025) is not less than 70, or more than half the total number in the square.

6. If a similar analysis is made of the proper motion diagram for other squares, we would obtain similar results. Even there we will find a similar displacement of the mean and a certain line of symmetry about which the motions of the stars are grouped. The difference is that the displacement of the mean takes place in different directions in different squares with a different amount and that the position of the line of symmetry, as well as corresponding dispersions, vary from one square to another. In the square GC_2 the line of symmetry is nearly parallel to the Milky Way. In other squares, for instance GB_5 and GE_{10}, it is nearly perpendicular to that plane.

In table 3 I give the mean values of the proper motions of the stars in galactic longitude and latitude for every square. The table embraces all stars (4182) brighter than the sixth visual magnitude for which the proper motions (in α and δ) are given in Boss' catalogue. Here 167 stars were excluded on account of large proper motion in u or v. The second column gives the mean proper motion $\bar{u}$ for each square ($\bar{u}$ is the proper motion in longitude multiplied by cos b). The third column gives the mean proper motion v in galactic latitude. The fourth column gives the total proper motion ($\bar{\mu}=\sqrt{\bar{u}^2+\bar{v}^2}$). The fifth column gives the direction, θ, of the motion reckoned from the u-axis toward the positive v-axis. The sixth column gives the number of stars in each square.

Regarding the values in this table we note the following:

First we observe the nearly pervading negative signs of v. We may conclude that the stars upon an average are moving in the direction of the south galactic pole. For the C and D squares of the Milky Way this apparent motion amounts on an average to $-0".0150$. We will find in the next chapter that this is a simple reflex effect of the motion of the sun against a point in the northern galactic hemisphere.

In longitude we find, however, that the positive and the negative motions are nearly represented in the same degree; yet not completely. If the proper motions in longitude of the 24 squares along the galactic equator (C and D squares) are added we find a mean displacement of $-0''.0023$. From the 20 B and E squares a mean displacement of $-0''.0025$ is obtained, hence of the same amount as for the C and D squares. Excluding the four pole squares, where u is poorly determined, we get a mean displacement of the stars in galactic longitude of $-0''.0024$. The mean error in this result is rather large. As far as we can trust in it, we may conclude that the galactic longitudes of the stars are on an average diminished by $0''.0024$ per year.

TABLE 3

Mean proper motions in galactic coördinates for all stars (4182) with $m<6.00$

	$\bar{u}$	$\bar{v}$	$\bar{\mu}$	θ	N		$\bar{u}$	$\bar{v}$	$\bar{\mu}$	θ	N
GA_1	+0″.0714	+0″.0092	+0″.0720	7°.3	49	GD_1	−0″.0034	−0″.0344	+0″.0346	263°.8	90
GA_2	−0.0588	−0.0153	+0.0608	194.6	62	GD_2	+0.0114	−0.0196	+0.0227	300.3	79
						GD_3	+0.0213	−0.0164	+0.0269	322.4	101
GB_1	−0.0150	+0.0275	+0.0313	118.5	80	GD_4	+0.0472	−0.0126	+0.0489	345.1	107
GB_2	+0.0384	+0.0062	+0.0389	9.2	73	GD_5	+0.0518	−0.0018	+0.0518	358.0	115
GB_3	+0.0146	+0.0048	+0.0154	18.2	72	GD_6	+0.0230	−0.0012	+0.0230	357.0	170
GB_4	+0.0418	−0.0212	+0.0469	333.1	73	GD_7	−0.0004	−0.0031	+0.0031	270.0	144
GB_5	+0.0400	−0.0572	+0.0698	305.0	55	GD_8	−0.0138	−0.0130	+0.0190	223.2	166
GB_6	0.0000	−0.0413	+0.0413	270.0	69	GD_9	−0.0369	−0.0150	+0.0398	202.1	160
GB_7	−0.0394	−0.0404	+0.0564	225.8	57	GD_{10}	−0.0340	−0.0174	+0.0382	207.1	106
GB_8	−0.0380	−0.0194	+0.0427	207.0	54	GD_{11}	−0.0458	−0.0254	+0.0524	209.0	106
GB_9	−0.0577	−0.0164	+0.0600	195.9	52	GD_{12}	−0.0237	−0.0268	+0.0358	228.5	97
GB_{10}	−0.0346	+0.0096	+0.0359	164.5	78						
						GE_1	+0.0082	−0.0396	+0.0404	281.4	67
GC_1	−0.0122	−0.0053	+0.0133	203.5	94	GE_2	+0.0226	−0.0403	+0.0462	299.3	62
GC_2	+0.0163	−0.0048	+0.0170	343.6	135	GE_3	+0.0287	−0.0410	+0.0500	304.9	61
GC_3	+0.0296	−0.0086	+0.0308	343.8	98	GE_4	+0.0590	−0.0070	+0.0594	353.2	72
GC_4	+0.0294	0.0000	+0.0294	0.0	95	GE_5	+0.0392	+0.0061	+0.0397	8.8	74
GC_5	+0.0416	−0.0128	+0.0435	342.9	78	GE_6	+0.0140	+0.0140	+0.0198	45.0	75
GC_6	+0.0370	−0.0354	+0.0512	316.3	92	GE_7	−0.0316	+0.0092	+0.0329	163.8	60
GC_7	−0.0036	−0.0202	+0.0205	260.2	84	GE_8	−0.0594	+0.0020	+0.0594	178.1	75
GC_8	−0.0198	−0.0263	+0.0329	233.1	78	GE_9	−0.0466	−0.0050	+0.0469	186.1	60
GC_9	−0.0392	−0.0216	+0.0448	208.8	102	GE_{10}	−0.0334	−0.0382	+0.0507	228.9	81
GC_{10}	−0.0625	−0.0200	+0.0656	197.8	120						
GC_{11}	−0.0339	−0.0095	+0.0352	195.7	121	GF_1	+0.0414	−0.0154	+0.0442	339.6	52
GC_{12}	−0.0345	−0.0098	+0.0359	195.8	71	GF_2	−0.0250	+0.0225	+0.0336	138.0	60

The systematic variation of the u-values from square to square is evident. This is more clearly shown by a graphical representation of the values of $\bar{u}$ and $\bar{v}$. It is better to trace the mean displacement for each square on a sphere, but even the Flamsteed projection gives a rather good representation. In the following diagram the mean displacement of the stars is given for each square.

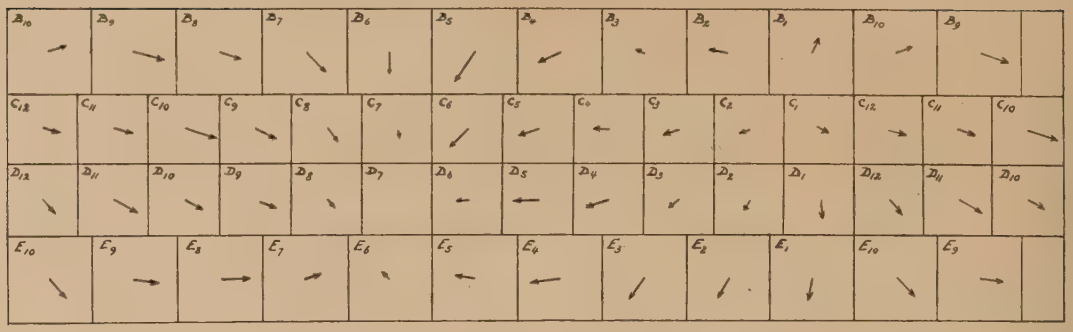

Diagram 3.—Mean Proper Motion—Different Squares.

It is evident from this diagram that the stars, through their proper motions, seem to approaching a point situated in the square GD_7 (Lepus) and to be receding from the square GC_1 (Hercules). It is the same phenomenon we have already found from the radial velocities, though the motions here have another geometrical significance. It may be described as follows: On an average, the stars seem to move along great circles on the sphere which intersect each other in the squares GD_7 and GC_1. The point from which the stars seem to radiate is called the *Apex*, the opposite point (in Lepus) the *Antiapex*. More accurate information regarding the position of these points shall be given in the next chapter.

To determine the position of the line of symmetry in different squares we have to compute the characteristics of the second order of the proper motions which give us the frequency distribution of the proper motions as referred to their means. We obtain for each square an elliptical frequency curve, the major axis of which determines the position of the line of symmetry. We thus get the results given in table 4.

The second column gives the number of stars in each square; the third column contains the angle φ of the line of symmetry that is supposed to pass through the mean of u and v; the fourth column gives the major axis (σ_1) of the frequency ellipse (which coincides with the line of symmetry), and the fifth column gives the minor axis (σ_2).

The elongated (elliptical) form of the frequency curve is well pronounced in most squares. In some squares, for instance GC_{12}, GD_6, and GD_{12}, the two axes are, however, nearly of the same size, giving a circular form to the frequency curve. If the ellipses of the different squares are drawn on a sphere, we find that the directions of the line of symmetry are not distributed at random. When all of them are prolonged they seem to pass nearly through two opposite points on the sphere in the constellations Gemini (GC_6) and Sagittarius (GD_{12}) or perhaps the neighboring squares (GD_6 and GC_{12}). These are approximately the same points where the dispersions in the radial velocities have their maximum values. The point of convergency in Sagittarius is called the *Vertex*, that in Gemini the *Antivertex*. A fuller discussion on the positions of these points will be given later.

TABLE 4

Frequency ellipse of the proper motions of all stars brighter than the sixth magnitude in different squares

	N	φ	σ_1	σ_2		N	φ	σ_1	σ_2
GA_1	49	−21.1	2.01	0.91	GD_1	90	−47.7	1.38	1.21
GA_2	62	− 9.3	2.55	1.00	GD_2	79	−15.5	1.31	0.69
					GD_3	101	+10.3	1.36	0.90
GB_1	80	+37.3	1.90	1.27	GD_4	107	− 8.5	1.92	0.95
GB_2	73	+28.4	2.00	1.08	GD_5	115	+ 7.9	1.22	0.68
GB_3	72	−12.3	1.80	0.76	GD_6	170	+16.6	0.89	0.81
GB_4	73	−48.4	1.71	1.06	GD_7	144	−11.5	0.99	0.66
GB_5	55	−90.0	1.47	0.88	GD_8	166	− 6.4	1.36	0.59
GB_6	69	+54.0	1.87	0.77	GD_9	160	− 2.2	1.50	0.85
GB_7	57	+20.8	2.18	1.69	GD_{10}	106	+ 6.8	1.69	0.88
GB_8	54	−15.2	1.63	0.90	GD_{11}	106	+42.0	1.46	0.84
GB_9	52	−51.1	1.75	1.24	GD_{12}	97	+27.1	1.29	1.00
GB_{10}	78	−53.0	1.47	1.04					
					GE_1	67	−43.5	1.86	1.06
GC_1	94	+15.1	1.26	0.73	GE_2	62	−21.0	2.14	0.99
GC_2	135	+ 2.1	1.52	0.64	GE_3	61	+ 9.7	1.95	1.33
GC_3	98	+ 5.6	1.33	0.78	GE_4	72	+23.7	1.98	1.04
GC_4	95	− 5.8	1.12	0.45	GE_5	74	+42.7	1.61	1.34
GC_5	78	−21.5	1.46	0.37	GE_6	75	−31.8	2.16	1.33
GC_6	92	−40.1	1.36	0.82	GE_7	60	− 6.8	1.95	0.97
GC_7	84	+ 3.2	1.16	0.77	GE_8	75	+ 2.2	2.06	1.14
GC_8	78	+21.7	1.22	0.51	GE_9	60	+26.1	1.92	1.27
GC_9	102	− 4.0	1.66	0.69	GE_{10}	81	+87.9	1.61	1.16
GC_{10}	120	+ 0.6	1.71	0.77					
GC_{11}	121	−13.0	0.96	0.64	GF_1	52	+ 2.1	1.43	1.17
GC_{12}	71	−13.1	1.51	1.20	GF_2	60	+23.1	1.63	1.01

7. If the characteristics of the third and fourth order of the frequency distribution are computed, the results will be of great interest. It may be that the number of stars in a square, brighter than the sixth magnitude, is rather low—the mean number is 87—and that the computation of the characteristics of the third and fourth order, the skewness and the excess, generally claims a greater number of individuals. If the value of these characteristics is very large, it is, however, possible to compute them with sufficient accuracy, even though the number of individuals is as low as 87. The corresponding mean error in the computation of the excess is equal to ± 0.07 and is small when the excess is nearly equal to unity or greater.

It is found that in all squares the excess is positive and large whereas the skewness, though also large, is alternatively positive or negative. The explanation of this positive and large value of the excess is given in L. M. II, 9.

8. *Summary.*—From the proper motions of the stars we find:

(1) The galactic latitude of the stars is diminishing in the mean at the galactic equator. The decrease amounts to $0''.0150$ per year.

(2) The galactic longitude of the stars is also diminishing in the mean. The accurate value of this decrease is still uncertain.

(3) If the mean proper motion of the stars in the different squares is calculated, it is found that this mean motion takes place along great circles that intersect one another (approximately) at two opposite points in the constellation of Hercules and Lepus (Apex and Antiapex).

(4) The frequency distribution of the proper motions is approximately represented by a frequency ellipse. The major axes in these ellipses, which may be called the lines of symmetry in the distribution, lie on great circles, which approximately intersect one another at two opposite points in the constellations Sagittarius and Gemini (vertex and antivertex).

(5) The excess of the frequency distribution is in all squares large and positive.

CHAPTER II

THE REAL MOTIONS OF THE STARS

A. Characteristics of the First Order. Apex

9. We will now show how the apparent motions of the stars described in the preceding chapter may be interpreted and transferred to absolute* motions in space.

The principal conclusions from the characteristics of the first order of the proper motions and of the radial velocities are explained through the motion of the sun; the conclusions from the characteristics of the second order depend on the real motions of the stars.

That the apparent motion of the stars as found from the radial velocities is really a reflex effect of the motion of the sun is a simple consequence of table 1. The stars in Hercules approach us with a velocity of about 4 Sir/st, the stars in Lepus recede from us with approximately the same velocity (I do not at present take into consideration the fact that the numerical value of the velocity is somewhat greater in the latter case than in the former). This may evidently be explained by supposing that the sun is moving toward Hercules (Apex) with a velocity of 4 Siriometers per stellar year. In other parts of the sky the relative radial motion of the stars must, evidently, be smaller than in the direction of the Apex and Antiapex. This relative motion is also shown by the numbers in table 1. It is found, too, that, at a distance of 90° from the Apex, the radial velocities are small or evanescent.

10. For a closer discussion of this hypothesis it is necessary to compute from the values of $\overline{W}$ obtained in the different squares the corresponding velocity of the sun and the corresponding coördinates of the Apex. The mathematical method for such a computation is given in chapter III. Using the 1986 stars treated in table 1 we thus find for the velocity (Ω) of the sun the value

$$\Omega = 4.009 \text{ Sir/st} = 18.99 \text{ km/sec.}$$

The coördinates of the Apex are found to be

$$L = 22°.87; \quad B = +23°.82$$

or

$$\alpha = 267°.32; \quad \delta = +30°.23.$$

Using these values we may compute the radial velocity (W_o) for every square, caused by this motion of the sun and compare it with the observed mean radial velocity $\overline{W}$. We then get the results of table 5.

* It is not intended to discuss the definition of 'absolute' coördinates. The 'absolute' motions here are referred to a system resting in the Galaxy.

TABLE 5

Mean radial velocity of the stars compared with the radial component of the velocity of the sun

	$\overline{W}$	$W_\circ$	$\overline{W}-W_\circ$		$\overline{W}$	$W_\circ$	$\overline{W}-W_\circ$
GA_1	−1.71	−1.85	+0.14	GD_1	−3.47	−3.11	−0.36
GA_2	−0.74	−1.34	+0.60	GD_2	−2.24	−2.89	+0.65
				GD_3	−1.59	−1.79	+0.20
GB_1	−3.33	−3.73	+0.40	GD_4	−0.48	−0.10	−0.38
GB_2	−2.78	−3.37	+0.59	GD_5	+3.14	+1.72	+1.42
GB_3	−1.37	−2.17	+0.80	GD_6	+4.54	+3.19	+1.35
GB_4	−0.44	−0.57	+0.13	GD_7	+4.94	+3.92	+1.02
GB_5	+1.10	+0.80	+0.30	GD_8	+4.08	+3.70	+0.38
GB_6	+1.84	+1.43	+0.41	GD_9	+2.80	+2.60	+0.20
GB_7	+0.88	+1.08	−0.20	GD_{10}	+0.73	+0.91	−0.18
GB_8	−1.30	−0.13	−1.17	GD_{11}	−0.39	−0.91	+0.52
GB_9	−1.15	−1.72	+0.57	GD_{12}	−1.81	−2.39	+0.58
GB_{10}	−3.97	−3.09	−0.88				
				GE_1	−0.88	−1.43	+0.55
GC_1	−3.99	−3.92	−0.07	GE_2	−1.30	−1.08	−0.22
GC_2	−3.35	−3.70	+0.35	GE_3	−0.23	+0.13	−0.36
GC_3	−2.68	−2.60	−0.08	GE_4	+1.53	+1.72	−0.19
GC_4	−1.22	−0.91	−0.31	GE_5	+2.46	+3.10	−0.64
GC_5	+0.84	+0.91	−0.07	GE_6	+2.48	+3.72	−1.24
GC_6	+2.21	+2.39	−0.18	GE_7	+3.23	+3.37	−0.14
GC_7	+3.72	+3.11	+0.61	GE_8	+2.05	+2.17	−0.12
GC_8	+4.08	+2.89	+1.19	GE_9	+1.97	+0.57	+1.40
GC_9	+1.68	+1.79	−0.11	GE_{10}	+0.72	−0.80	+1.52
GC_{10}	+0.17	+0.10	+0.07				
GC_{11}	−2.24	−1.72	−0.52	GF_1	+1.11	+1.34	−0.23
GC_{12}	−2.50	−3.19	+0.69	GF_2	+2.11	+1.85	+0.26
				Mean			+0.164

The second column gives the observed $\overline{W}$, the third column the radial velocity $W_\circ$ which would arise for the stars in the different squares from the motion of the sun toward the Apex given above.

We find that the systematic variation in $\overline{W}$ is precisely reproduced in the values of $W_\circ$. There is no doubt that the mean values of the radial velocities are substantially interpreted by the motion of the sun toward the Apex. One fact seems to be left unexplained, namely, that the signs of the difference $\overline{W}-\overline{W}_\circ$ are generally positive, so that the observed mean radial velocity is generally greater than that caused by the motion of the sun. The discussion of this result is given later in this chapter.

11. Before we can discuss the accordance of the motion of the sun with the observed proper motions of the stars, we must first compute the velocity com-

ponents, in the galactic coördinate system, of the sun as derived from the radial velocities. Let X, Y, Z be these components. Then we derive from the values of $\overline{W}$ given in table 1

$$X = +3.371 \text{ Sir/st}$$
$$Y = +1.447 \text{ Sir/st}$$
$$Z = +1.619 \text{ Sir/st}$$

From these velocity components we obtain the value of the total velocity (Ω) of the sun and the position of the Apex given in §10.

Instead of considering the sun as moving relative to the stars we may, like the observer, take the sun as the fixed origin of the coördinates and regard the system of the stars as moving relative to the sun. We then consider the stars as members of a stream having velocity components equal to X, Y, Z with inverse sign. We denote these stream components by $U_\circ''$, $V_\circ''$, and $W_\circ''$, so that

$$U_\circ'' = -3.371 \text{ Sir/st}$$
$$V_\circ'' = -1.447 \text{ Sir/st}$$
$$W_\circ'' = -1.619 \text{ Sir/st}$$

It is then possible to compute the stream components $U_\circ$, $V_\circ$, $W_\circ$ in a system of coördinates (K_3), having the Z-axis directed toward the center of gravity of a square, the positive X- and Y-axes directed toward increasing galactic longitude and latitude. In table 6 I give the values of these components.

The components $U_\circ$ and $V_\circ$ must be proportional to the observed proper motion values of u and v so far as those arise from the motion of the sun. More precisely we must have

$$\vartheta_1 U_\circ = \overline{u} \text{ and } \vartheta_1 V_\circ = \overline{v}$$

where ϑ_1 is equal to the mean value of $1/r$, r being the distance of the individual stars in the square. We may also write that

$$\vartheta_1 \omega_\circ = \vartheta_1 \sqrt{U_\circ^2 + V_\circ^2} \quad = \sqrt{\overline{u}^2 + \overline{v}^2} = \overline{\mu}$$

μ being the total proper motion of a star.

Writing

$$U_\circ = \omega_\circ \cos \theta_\circ; \quad V_\circ = \omega_\circ \sin \theta_\circ$$
$$\overline{u} = \overline{\mu} \cos \theta; \quad \overline{v} = \overline{\mu} \sin \theta$$

we find that $\theta_\circ$ determines the direction of the stream velocity for each square and θ the direction of the observed mean proper motion. We then ought to have, approximately,

$$\theta = \theta_\circ$$

TABLE 6

Components of the stream velocity for different squares

	$U_\circ$	$V_\circ$	$W_\circ$		$U_\circ$	$V_\circ$	$W_\circ$
GA_1	+3.37	+1.14	−1.85	GD_1	−0.52	−2.48	−3.11
GA_2	−3.37	−1.71	−1.34	GD_2	+1.36	−2.42	−2.89
				GD_3	+2.88	−2.14	−1.79
GB_1	−0.33	+1.44	−3.73	GD_4	+3.63	−1.70	−0.10
GB_2	+1.88	+1.09	−3.37	GD_5	+3.41	−1.23	+1.72
GB_3	+3.37	−0.43	−2.17	GD_6	+2.27	−0.85	+3.19
GB_4	+3.58	−1.72	−0.57	GD_7	+0.52	−0.66	+3.92
GB_5	+2.42	−3.10	+0.80	GD_8	−1.36	−0.72	+3.70
GB_6	+0.33	−3.73	+1.43	GD_9	−2.88	−1.00	+2.60
GB_7	−1.88	−3.38	+1.08	GD_{10}	−3.63	−1.44	+0.91
GB_8	−3.37	−1.85	−0.13	GD_{11}	−3.41	−1.91	−0.91
GB_9	−3.58	−0.57	−1.72	GD_{12}	−2.27	−2.29	−2.39
GB_{10}	−2.42	+0.81	−3.09				
				GE_1	−0.33	−3.73	−1.43
GC_1	−0.52	−0.66	−3.92	GE_2	+1.88	−3.38	−1.08
GC_2	+1.36	−0.72	−3.70	GE_3	+3.37	−1.85	+0.13
GC_3	+2.88	−1.00	−2.60	GE_4	+3.58	−0.57	+1.72
GC_4	+3.63	−1.44	−0.91	GE_5	+2.42	+0.81	+3.10
GC_5	+3.41	−1.91	+0.91	GE_6	+0.33	+1.44	+3.72
GC_6	+2.27	−2.29	+2.39	GE_7	−1.88	+1.09	+3.37
GC_7	+0.52	−2.48	+3.11	GE_8	−3.37	−0.43	+2.17
GC_8	−1.36	−2.42	+2.89	GE_9	−3.58	−1.72	+0.57
GC_9	−2.88	−2.14	+1.79	GE_{10}	−2.42	−3.10	−0.80
GC_{10}	−3.63	−1.70	+0.10				
GC_{11}	−3.41	−1.23	−1.72	GF_1	+3.37	−1.71	+1.34
GC_{12}	−2.27	−0.85	−3.19	GF_2	−3.37	+1.14	+1.85

We may in this manner, simply by comparing θ and $\theta_\circ$, state the consistency of the proper motions of the stars with the hypothesis regarding the motion of the sun. By further comparing $\omega_\circ$ and $\bar{\mu}$ we are able to determine ϑ_1 and hence the mean parallax ($\bar{\pi}$) of the stars and also their harmonic mean distance (H). We have indeed

$$\bar{\pi} = 0''.206265\ \vartheta_1$$
$$H = 1/\vartheta_1$$

In table 7 I give in the second column, θ, in the third, $\theta_\circ$, in the fourth, $\bar{\mu}$, in the fifth, $\omega_\circ = \sqrt{U_\circ^2 + V_\circ^2}$, in the sixth, the mean parallax ($\bar{\pi}$), in the seventh, the harmonic mean distance (H) for each square.

TABLE 7

Streamdirection (θ and θ_o), streamvelocity ($\bar{\mu}$, ω_o), meanparallax ($\bar{\pi}$) and harmonic mean distance (H) of stars brighter than the sixth magnitude

	θ	θ_o	$\bar{\mu}$	ω_o	$\bar{\pi}$	H		θ	θ_o	$\bar{\mu}$	ω_o	$\bar{\pi}$	H
GA_1	7°	19°	0″.072	3.56	0″.020	10	GD_1	264°	258°	0″.035	2.53	0″.014	15
GA_2	195	207	0.061	3.78	0.016	13	GD_2	300	299	0.023	2.78	0.008	25
							GD_3	322	324	0.027	3.59	0.008	28
GB_1	118	103	0.031	1.48	0.021	10	GD_4	345	335	0.049	4.01	0.012	17
GB_2	9	30	0.039	2.17	0.018	12	GD_5	358	340	0.052	3.62	0.014	14
GB_3	18	353	0.015	3.40	0.004	45	GD_6	357	340	0.023	2.42	0.010	22
GB_4	333	334	0.047	3.97	0.012	17	GD_7	270	308	0.003	0.84	0.004	56
GB_5	305	308	0.070	3.93	0.018	12	GD_8	223	208	0.019	1.54	0.012	17
GB_6	270	275	0.041	3.75	0.011	19	GD_9	202	199	0.040	3.05	0.013	16
GB_7	226	241	0.056	3.86	0.015	14	GD_{10}	207	202	0.038	3.90	0.010	21
GB_8	207	209	0.043	3.85	0.011	19	GD_{11}	209	209	0.052	3.90	0.013	15
GB_9	196	189	0.060	3.62	0.017	12	GD_{12}	228	225	0.036	3.22	0.011	19
GB_{10}	164	161	0.036	2.55	0.014	15							
							GE_1	281	265	0.040	3.75	0.011	19
GC_1	204	232	0.013	0.84	0.016	13	GE_2	299	299	0.046	3.86	0.012	17
GC_2	344	332	0.017	1.54	0.011	19	GE_3	305	331	0.050	3.85	0.013	16
GC_3	344	341	0.031	3.05	0.010	20	GE_4	353	351	0.059	3.62	0.016	13
GC_4	0	338	0.029	3.90	0.008	27	GE_5	9	19	0.040	2.55	0.016	13
GC_5	343	331	0.044	3.90	0.011	19	GE_6	45	77	0.020	1.48	0.013	15
GC_6	316	315	0.051	3.22	0.016	13	GE_7	164	150	0.033	2.17	0.015	14
GC_7	260	282	0.020	2.53	0.008	25	GE_8	178	187	0.059	3.40	0.018	12
GC_8	233	241	0.033	2.78	0.012	17	GE_9	186	206	0.047	3.97	0.012	17
GC_9	209	216	0.045	3.59	0.012	17	GE_{10}	229	232	0.051	3.93	0.013	16
GC_{10}	198	205	0.066	4.01	0.016	13							
GC_{11}	196	200	0.035	3.62	0.010	21	GF_1	340	333	0.044	3.78	0.012	18
GC_{12}	196	200	0.036	2.42	0.015	14	GF_2	138	161	0.034	3.56	0.009	22

We may for the sake of fulness repeat the signification of the headings.

θ = the mean direction of the proper motions of the stars in a square.

θ_o = the direction of the stream velocity = the inverse of the component of the velocity of the sun.

$\bar{\mu}$ = the mean total angular motion of the stars.

ω_o = the component of the stream velocity perpendicular to the line of sight.

$\bar{\pi}$ = the mean parallax of the stars to the sixth magnitude for each square.

H = the harmonic mean distance of the stars expressed in Siriometers.

A comparison between θ and θ_o confirms the fact that the mean direction of the proper motions of the stars is approximately the same as the direction of the stream velocity. The differences obtained are well explained through the mean errors in θ, which may be smaller when the number of stars with known proper motions increases. From the mean total proper motions and the stream velocity ω_o we derive the mean parallaxes in the sixth column.

We find from the table that the mean parallaxes of the stars brighter than the sixth magnitude are approximately the same for all squares and are nearly independent of the galactic latitude. In the *GA* squares (*GA*$_1$ and *GA*$_2$), about the northern galactic pole, we have $\overline{\pi}=0''.0180$. In the *GF* squares about the southern galactic pole, $\overline{\pi}=0''.0105$. In the *GC* squares, north of the galactic equator we have $\overline{\pi}=0''.0121$. Or more fully

GA	squares	$\overline{\pi} = 0''.0180$
GB	squares	$\overline{\pi} = 0.0141$
GC	squares	$\overline{\pi} = 0.0121$
GD	squares	$\overline{\pi} = 0.0108$
GE	squares	$\overline{\pi} = 0.0139$
GF	squares	$\overline{\pi} = 0.0105$

Taking the northern and the southern squares of the same latitude together we get

AF	squares	$\overline{\pi} = 0''.0142$
BE	squares	$\overline{\pi} = 0.0140$
CD	squares	$\overline{\pi} = 0.0114$

We find from these numbers that a small increase in $\overline{\pi}$ is suggested for stars outside the Milky Way.

The mean parallax of the stars brighter than the sixth magnitude is

$$\overline{\pi}=0''.0127.$$

From $\overline{\pi}$ (or ϑ_1) we may also derive the harmonic mean distance of the stars which are given in the seventh column. It varies, as is to be seen, between 10 Siriometers and 56 Siriometers and is, as well as $\overline{\pi}$, nearly independent of the galactic latitude.

GA	squares	$H=11.5$
GB	squares	$H=17.5$
GC	squares	$H=18.2$
GD	squares	$H=22.1$
GE	squares	$H=15.2$
GF	squares	$H=20.0$

The mean value of these numbers is

$$H=18.2 \text{ Siriometers.}$$

Though it is possible in this manner to derive the harmonic mean distance of the stars, we do not know the arithmetic mean distance ($\bar{r}$) of the stars. We know that $\bar{r}$ is greater than H, (the arithmetic mean of positive quantities is always greater than their harmonic mean) but for determining its value more accurately, we ought to know the distribution of the stars in space as well as the dispersion in their absolute magnitude. The corresponding problems are discussed in the next section.

12. From the proper motions of the stars we are unable to determine directly the value of the velocity of the sun; that can only be done from the proper motions of those stars for which the distance from the sun are known. Such stars are called parallax stars. To the autumn of 1922 the number of these stars was 654. Excluding 8 stars, which have considerably large velocities, I have obtained from the remaining 646, for the velocity of the sun, the value:

$$\Omega = 3.764 \text{ Sir/st} = 17.83 \text{ km/sec}$$

or nearly (somewhat smaller) the same value as from the radial velocities. This value, as obtained from a greater number of stars, may be considered more accurate.

For the coördinates of the Apex I get, from the parallax stars, the values:

$$L = 23°.40; \quad B = +22°.06$$

or

$$\alpha = 269°.42; \quad \delta = +30°.09$$

These coördinates do not differ very much from those found from the radial velocities.

13. Though the velocity of the sun may not be directly derived from the proper motions of the stars, these may, however, be used for the determination of the coördinates of the Apex. The prolongation of the small arcs described by the stars on the sky must on an average interesect one another at the Apex and Antiapex. The coördinates of the Apex obtained in this manner are the following:

$$L = 26°.12 \qquad B = +21°.45$$

or

$$\alpha = 271°.12 \qquad \delta = +32°.24.$$

The value of L deviates some three degrees from that obtained from the 1986 radial velocity stars. The value of B is 1° smaller. The method for determining the Apex from the proper motions is shown in chapter III.

14. The position of the Apex may be determined from any group of stars for which the proper motions are known. Such determinations have, indeed, been made many times since Herschel's first solution from 14 proper motion stars in the year 1783, when he proposed to explain the proper motions of the stars as a reflex effect of the motion of the sun. The coördinates of the Apex

determined from different solutions, have generally agreed tolerably well. The astronomers usually give the position in equatorial coördinates (α, δ) and it has been found that the values of the right ascension of the Apex generally deviate only a little from each other, the mean value being about $\alpha = 270°$, whereas the declination may show rather great variations. If the position is given in galactic coördinates, this variation appears in another form and finds an easy explanation.

In the following table I have given some determinations of the coördinates of the Apex obtained for groups of stars of different spectral type and groups of stars having different apparent magnitude.

TABLE 8

Galactic coördinates of Apex obtained from different groups of stars

		Number of stars	L	B
(1)	B−stars	694	26°.7	+17°.3
(2)	A−stars	1281	18.6	+22.8
(3)	F−stars	508	17.8	+20.4
(4)	G−stars	379	29.7	+28.6
(5)	K−stars	1135	37.7	+17.8
(6)	M−stars	190	31.5	+20.6
(7)	$m =$ 5.0 to 5.9	454	20.	+22.
(8)	$m =$ 6.0 to 6.9	1003	29.	+16.
(9)	$m =$ 7.0 to 7.9	1239	35.	+17.
(10)	$m =$ 8.0 to 8.9	811	37.	+24.
(11)	$m =$ 8.6	276	55.	+23.
(12)	$m =$ 11.1	203	69.	+24.

The groups (1) to (6), containing stars brighter than the sixth magnitude of different spectral types, have been treated by Gyllenberg and myself in *L. M.* 92. The groups (7) to (10), containing proper motion stars determined in Greenwich, have been treated by Dyson and Thackeray in *M. N.* LXV and the groups (11) and (12) have been examined by Comstock in *A. J.* 655.

It is evident from this table that the galactic latitude of the Apex varies slightly for the different groups. The values vary with small dispersion about the value $B = 23°$ obtained from the radial velocities and the proper motions of all stars brighter than the sixth magnitude. In galactic longitude the variations are, however, greater. Notable is the seemingly systematic increase of L with increasing apparent magnitude of the stars in the groups (7) to (12). This increase of L corresponds to the increase of δ to which Dyson and Thackeray have called attention in their memoir. A glance at plate 1 of the *Introduction to Stellar Statistics* shows how the declination circles at this part of the sky are nearly parallel circles in the galactic system of coördinates. It is, however, easily understood why the longitude of the Apex may vary from one group of stars to another, whereas the latitude remains nearly constant. This implies only that one group

of stars moves in a different direction in the galactic plane from another group, but that the mean motions of the groups are parallel to this plane. A systematic investigation of this question will give important contributions to our knowledge regarding the movement of the stars in the Galaxy.

Summary

(1) From the characteristics of the first order of the radial velocities (W) and the proper motions (u and v) of the stars we have found that these values are in their main features explained through a motion of the sun. From the radial velocities in different squares (1986 stars) we have concluded that the corresponding velocity of the sun amounts to 4.0 Siriometers per stellar year (4 Astronomical units per year) or 19.0 kilometers per second. This result is nearly confirmed by the motion of known parallax stars (646).

(2) The sun is moving toward a point (Apex), the position of which as determined from 1986 radial velocity stars and 4182 proper motion stars brighter than the sixth magnitude and 646 parallax stars, is found to be (in the mean)

$$L = 24^\circ.13 \qquad B = +22.44$$

or

$$\alpha = 269^\circ.28 \qquad \delta = +30^\circ.85$$

(3) The position of the Apex may be found from any group of stars. Different groups of stars give nearly the same latitude of the Apex but rather varying values of the longitude, this latter because of a motion of the groups in different directions but all parallel to the galactic plane.

(4) The mean parallax of all stars brighter than the sixth magnitude is equal to $0''.0127$. Their harmonic mean distance is 18 Siriometers.

B. Characteristics of the Second Order

15. Although the laws controlling the absolute velocities of the stars may be completely studied with the help of the proper motions and the radial velocities of the stars, it is evident that the discussion of these velocities would be simplified if we directly knew the three components of the velocity of the stars. This is indeed possible for the parallax stars, supposing that not only the parallax of the stars but also their radial velocity and their proper motions in galactic longitude and latitude are determined. When the distance is known we may directly deduce, from the proper motions, their velocity components perpendicular to the line of sight. The third component is given by the radial velocity. We then have the three velocity components in the system K_3 of coördinates and obtain, through a simple turning of axes, the velocity components in the fixed system

(K_2) of coördinates coinciding with the Galaxy. Unfortunately, at present we know only the parallaxes for a rather small number of stars, and these stars are moreover selected in a special manner so that we are not able to discover their general characteristic features in the stellar system. The discussion for parallax stars being quite simple, I shall now take up the consideration of the 646 parallax stars mentioned in §12.

We suppose that the velocities of these stars are all reduced to the galactic system (K_2) of coördinates. We then know for each of the 646 stars the velocity components in this system, which we may denote by

$$U'', \; V'', \; W''$$

With these coördinates we form a frequency distribution of three dimensions as we did in §11 for a frequency distribution of two dimensions from the proper motions of the stars. Studying this frequency distribution of three dimensions according to the general methods of mathematical statistics, we first compute the means values $U_\circ''$, $V_\circ''$, and $W_\circ''$ of the three velocity components and obtain

$$\begin{aligned} U_\circ'' &= -3.201 \\ V_\circ'' &= -1.385 \\ W_\circ'' &= -1.413 \end{aligned}$$

This is nothing but the velocity components, with inverse signs, of the sun relative to the parallax stars. They nearly coincide with the values of these components obtained in §11 from radial velocity stars. If $U_\circ''$, $V_\circ''$, and $W_\circ''$ are subtracted from the given velocity components U'', V'', W'' we obtain the absolute velocity components of the stars:

$$U''-U_\circ'', \quad V''-V_\circ'', \quad W''-W_\circ''.$$

Studying the frequency distribution of these components the immediate task is to compute the moments of the second order which give the principal characteristics of the frequency function. As far as the investigation is limited to these characteristics, we find that the surfaces of equal frequency in the distribution of the velocities are ellipsoids, the axes of which are proportional to the semiaxes of a certain surface, the *velocity ellipsoid*, the equation of which is given by

$$\frac{x^2}{\sigma_1^2}+\frac{y^2}{\sigma_2^2}+\frac{z^2}{\sigma_3^2}=1$$

The semiaxes σ_1, σ_2, σ_3 of this ellipsoid as determined from the 646 parallax stars, have the following values:

$$\begin{aligned} \text{I} &: \sigma_1 = 3.410 \text{ Sir/st} = 16.15 \text{ km/sec} \\ \text{II} &: \sigma_2 = 4.088 \text{ Sir/st} = 19.37 \text{ km/sec} \\ \text{III} &: \sigma_3 = 5.880 \text{ Sir/st} = 27.86 \text{ km/sec} \end{aligned}$$

Their direction, defined by their galactic longitude and latitude, is the following:

$$\begin{aligned} \text{I} &: L_1 = 233^\circ.06; \quad B_1 = +83^\circ.07 \\ \text{II} &: L_2 = \ 71.36; \quad B_2 = +\ 7.65 \\ \text{III} &: L_3 = 341.29; \quad B_3 = +\ 2.84 \end{aligned}$$

The smallest axis (σ_1) is somewhat smaller than the velocity of the sun and is approximately directed toward the pole of the Galaxy.

The other two axes lie almost in the galactic plane. The greatest axis (σ_3) is directed toward the point

$$L_3 = 341^\circ.29; \quad B_3 = +2^\circ.84$$

situated in Ophiuchus (or Sagittarius). The opposite direction of this axis (III) is toward a point between the constellations Gemini and Orion.

When the velocity ellipsoid is known we may calculate the frequency distribution of all values of the velocity components. The total frequency in any direction (the total number of stars moving in this direction) is proportional to the third power of the radius vector of the velocity ellipsoid (drawn from the center of the ellipsoid) in this direction. We conclude that the larger number of the parallax stars move in the direction of the axis III (or in the opposite direction). The point (L_3, B_3) in Sagittarius is called the vertex. The opposite point in Gemini, is called antivertex. The number of stars moving in the direction of the three axes σ_1, σ_2, σ_3 have the proportion

$$\sigma_1^3 : \sigma_2^3 : \sigma_3^3 : = (3.41)^3 : (4.09)^3 : (5.88)^3 = 1 : 1.87 : 5.55$$

The number of stars moving toward the vertex or antivertex is more than five times the number of stars moving toward the pole of the Galaxy (the direction of axis I), and three times the number of stars moving toward axis II (in Cassiopeia).

We note that the vertex and antivertex nearly coincide with the points determined in §3 for the maximum values of the dispersion in the radial velocities. The dispersions of the velocities in different directions, are, indeed, themselves determined through an ellipsoid

$$\sigma_1^2 X^2 + \sigma_2^2 Y^2 + \sigma_3^2 Z^2 = 1$$

which may be called the 'moment ellipsoid' (we may also use the name, 'dispersion ellipsoid'), the axes of which have the inverse value of the axes of the velocity ellipsoid. The radius vector of the moment ellipsoid in any direction is equal to the inverse value of the dispersion in the same direction. (For proof of these theorems see Chap. III.) The dispersions in the directions I, II, and III are equal to σ_1, σ_2, and σ_3, respectively.

16. The results of the preceding paragraph were based on the discussion of the motions of 646 parallax stars. I have already remarked that from these stars,

which are specially selected as bright stars and stars with great proper motion, it is not possible to gain an adequate idea of the characteristic features in the stellar system.

It is, however, possible to study the absolute motions of the stars without knowing their distances from us. This can be done, for instance, for all stars for which the radial velocities are determined, supposing only that their number is sufficiently great. We may, then, for each square, or any other spherical area, compute the mean value of the radial velocities and the dispersion. From the mean radial velocities ($\overline{W}$) we compute, as has been shown in §11, the mean velocity components of the stars in the galactic system of coördinates (K_2), which means, with inverse signs, the velocity components of the sun. In like manner we may from the dispersions of W in the different squares, using the method of least squares, compute the moments of the second order in the same system (K_2) and hence the parameters of the velocity ellipsoid, i. e., the semiaxes of the ellipsoid and their direction in space.

The 1986 stars, with known radial velocities, which were used in §10 for determining the velocity of the sun, were treated in this manner, and the following values of the axes of the velocity ellipsoid obtained:

$$\begin{aligned} \text{I} &: \sigma_1 = 3.294 \text{ Sir/st} = 15.61 \text{ km/sec} \\ \text{II} &: \sigma_2 = 2.835 \text{ Sir/st} = 13.43 \text{ km/sec} \\ \text{III} &: \sigma_3 = 4.125 \text{ Sir/st} = 19.54 \text{ km/sec} \end{aligned}$$

The direction of the axes is the following:

$$\begin{aligned} \text{I} &: L_1 = 270^\circ.00; \quad B_1 = +72^\circ.77 \\ \text{II} &: L_2 = 69^\circ.58; \quad B_2 = -16^\circ.17 \\ \text{III} &: L_3 = 341^\circ.20; \quad B_3 = -5^\circ.71 \end{aligned}$$

Hence we also obtain here one axis nearly directed toward the north pole of the Galaxy and two axes, approximately, in the galactic plane. The greatest axis is also here directed toward Ophiuchus-Sagittarius. Of the two minor axes we find, however, that not the least but the mean axis is directed toward the galactic pole. The difference between the axes is, however, small and may be due to temporary errors. We may observe that even the 1986 radial velocity stars are not chosen at random, though the method of selection is not altogether the same as with the parallax stars. Using the mean of the axes σ_1 and σ_2, we find that the number of stars moving in the direction of the three axes is in the following proportion:

$$(14.52)^3 : (14.52)^3 : (19.54)^3 = 1 : 1 : 2.44$$

so that more than twice the number of stars move toward the vertex (III) than in the direction of the other axes.

Hence we find that the vertex is approximately the same here as for the parallax stars, though the proportion of stars moving toward the vertex is not so great here. On the whole the axes of the ellipsoid are smaller here than for the parallax stars.

17. Even the proper motion stars may be used to determine the velocity ellipsoid of the stars, although a complete determination of the parameters cannot be performed. It is, however, possible in this manner to determine the direction of the axes and their relative magnitude. In order to obtain their absolute values we make use of the results obtained from the radial velocity stars.

We indicated in §§5, 6, how it is possible to compute from the proper motions u and v a proper motion ellipse in which the square of the radius vector in any direction is proportional to the number of stars moving in the same direction. Further, from the velocity ellipsoid we may compute the distribution of the components of the velocities of the stars in a square perpendicular to the line of sight, and deduce the corresponding velocity ellipse, which is, indeed, equal to the projection of the velocity ellipsoid on the square. Comparing now the surface of this velocity ellipse with the surface of the proper motion ellipse, we obtain the value of

$$\vartheta_2 = M\left(\frac{1}{r^2}\right)$$

which enables us to reduce the values of the parameters of the velocity ellipsoid, as obtained from the proper motions, to absolute measures. For a fuller description of this point I refer to Chapter III.

The values of ϑ_2 for the different squares are the following:

	ϑ_2		ϑ_2		ϑ_2		ϑ_2
GA_1	0.0089	GC_1	0.0048	GD_1	0.0081	GE_1	0.0078
GA_2	0.0100	GC_2	0.0042	GD_2	0.0037	GE_2	0.0082
GB_1	0.0121	GC_3	0.0040	GD_3	0.0044	GE_3	0.0098
GB_2	0.0094	GC_4	0.0018	GD_4	0.0064	GE_4	0.0088
GB_3	0.0049	GC_5	0.0039	GD_5	0.0040	GE_5	0.0099
GB_4	0.0073	GC_6	0.0058	GD_6	9.0038	GE_6	0.0134
GB_5	0.0072	GC_7	0.0044	GD_7	0.0034	GE_7	0.0080
GB_6	0.0062	GC_8	0.0029	GD_8	0.0037	GE_8	0.0094
GB_7	0.0134	GC_9	0.0047	GD_9	0.0051	GE_9	0.0094
GB_8	0.0056	GC_{10}	0.0055	GD_{10}	0.0055	GE_{10}	0.0082
GB_9	0.0100	GC_{11}	0.0029	GD_{11}	0.0055	GF_1	0.0065
GB_{10}	0.0072	GC_{12}	0.0097	GD_{12}	0.0062	GF_2	0.0067
Mean	0.0085		0.0046		0.0050		0.0088

The mean value for all squares is

$$\vartheta_2 = 0.006723$$

Remarkable is the pronounced dependence of the galactic latitude for the values of ϑ_2. For the C and D squares about the galactic equator its value is 0.0048, with rather small dispersion, whereas in the squares with higher latitude we get $\vartheta_2 = 0.0086$, nearly double the value. The value of ϑ_1 giving the mean

value of $1/r$, was, found in §11, somewhat dependent upon the galactic latitude. The effect of the distance is, however, evidently greater when the mean value of $1/r^2$ is to be calculated. That ϑ_2 is smaller at the Milky Way than toward the pole, when all stars brighter than the sixth magnitude are taken into consideration, shows that the absolutely bright stars are more concentrated in the galactic plane, so that, when the mean value of $1/r^2$ is to be taken for all stars apparently brighter than the sixth magnitude, we have to penetrate farther into the Milky Way than toward the poles, in which direction the absolutely bright stars are more sparse. The values of ϑ_2, as well as of ϑ_1, play a great role in the next section, where the distribution of the stars in space is to be discussed.

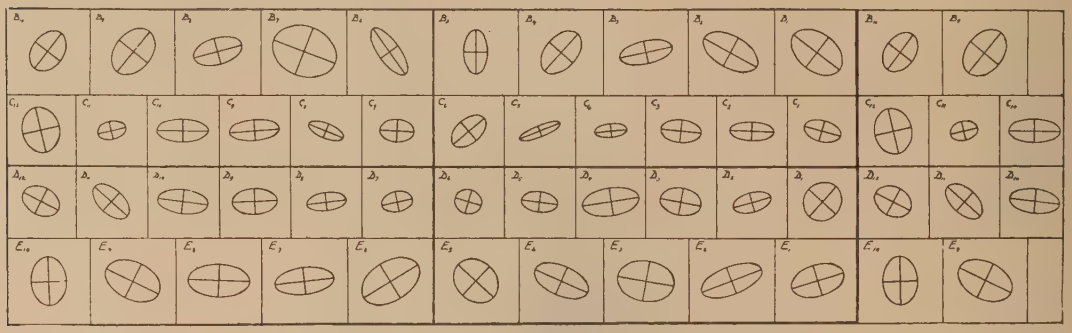

Diagram 4.—Velocity Ellipses (Observed)—Different Squares.

In diagram 4 I give the proper motion ellipses in the different squares. We may state that the major axes (the lines of symmetry), prolonged, pass approximately, through the vertex and antivertex to which we have already called attention. The relative dimensions (surfaces) of the ellipses are proportional to the values of ϑ_2, being small at the Milky Way and large at great latitudes. The velocity ellipses of the components for each square are given in diagram 5.

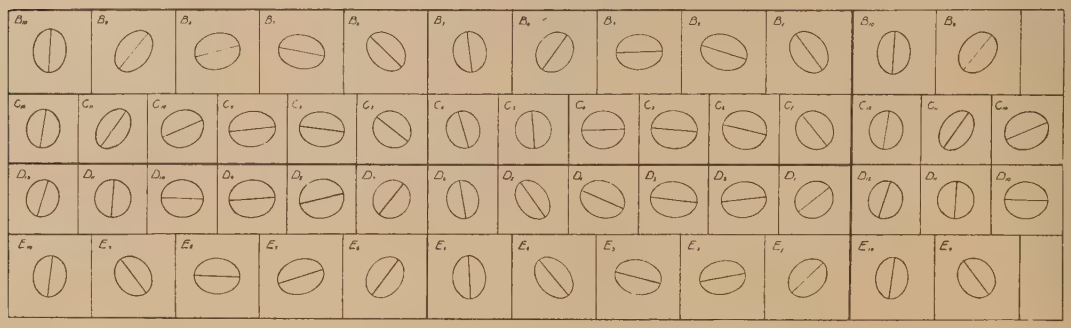

Diagram 5.—Velocity Ellipses (Theoretical)—Different Squares.

A comparison between diagrams 4 and 5 illustrates directly the variation in ϑ_2 given in the table on page 27.

18. The value of ϑ_2 being given we are able to completely determine the velocity ellipsoid from the proper motions of the stars. In this manner we obtain the following values of the semiaxes of the ellipsoid

$$\sigma_1 = +2.546 \text{ Sir/st} = 12.06 \text{ km/sec}$$
$$\sigma_2 = +3.185 \text{ Sir/st} = 15.09 \text{ km/sec}$$
$$\sigma_3 = +4.744 \text{ Sir/st} = 23.42 \text{ km/sec}$$

The direction of the axes is given by

$$L_1 = 52°.86 \qquad B_1 = +75°.84$$
$$L_2 = 67.76 \qquad B_2 = -13.40$$
$$L_3 = 339.01 \qquad B_3 = -3.73$$

C. Expansion of the Stars. Motion of the Invariable Plane.

19. *The expansion of the stars.*—We have found in §2 that the mean value of the radial velocities of the stars in a square has upon an average a positive value and that we therefore may assume that the stars in the mean are receding from the sun. The conclusion is somewhat uncertain since in table 1, from which the result was obtained, positive as well as negative values of $\overline{W}$ occur and both nearly in the same frequency. The mean error in the mean value of $\overline{W}$ is therefore rather great. We know, however, from §10 that these variations in $\overline{W}$, at least in the main, depend on the motion of the sun and that approximately,

$$\overline{W} = W_\odot$$

where $W_\odot$ is the component with inverse sign of the motion of the sun in the line of sight. We must therefore directly compare the values of $\overline{W}$ and $W_\odot$ in the different squares and compute the differences $\overline{W} - W_\odot$ for each square. This is done in table 5, which gives $\overline{W}$, $W_\odot$, and $\overline{W} - W_\odot$ for each one of the 48 galactic squares.

Looking at these differences we find it confirmed, though negative values occur, that the difference between $\overline{W}$ and $W_\odot$ is generally positive. The mean value is

$$+0.193 \text{ Sir/st}$$

with a mean error of ± 0.091 Sir/st.

We may therefore consider it as rather well established that the stars upon an average recede from us with a velocity of, approximately, 0.2 Siriometers per stellar year or one kilometer per second. This fact was first suggested by Campbell and more recently stated by Gyllenberg and others. How is this "expansion" of the stars to be explained?

20. The question is of recent date. First pointed out by Campbell (1912), so far no satisfactory explanation has been found. Whether the expansion is con-

sidered as real—depending on a real motion of the stars out of the center of the galaxy—or only apparent, it appears equally mystical.

If the stars really are moving out from our sun (or, what is nearly the same, from the center of the galaxy), we could give as a plausible explanation the possibility that our Galaxy is a spiral nebula and that the stars are moving out from the center along the spiral branches. Opposed to this explanation is the fact that the values of $\overline{W}-W_\circ$ in table 5 nowhere show such a distribution of the values to be compatible with such an hypothesis. Nor is there a pronounced systematic variation of the galactic latitude.

If, on the other hand, the expansion is only apparent, we may reflect on the following hypotheses in explanation of the phenomenon.

(*a*) Systematic errors may be suspected in the measures of the displacement of the spectral lines. Campbell, after thorough examination, excludes this explanation. It is sufficient to state that the mean error of a radial velocity amounts to some few hundredths of a Siriometer per stellar year and the expansion is 10 to 20 times this value.

(*b*) An hypothesis that deserves more attention, and which has also been discussed by Campbell, is that the displacement might depend upon the great density in the absorbing atmosphere of those stars for which the expansion is most pronounced. This possibility is based on the results obtained by Campbell as well as by Gyllenberg; namely, that the expansion is greatest for the brightest stars—types B and M—whereas it is evanescent or negative for other types. Newer material regarding the radial velocities does not confirm this conclusion; nor would this conclusion, even if stated, import a confirmation of the hypothesis in question. For, even if the B and the M stars (of the giant type) be regarded as the greatest stars, they do not represent stars of the same nor of the highest densities. We must therefore abandon this explanation.

(*c*) Another explanation of the expansion of the stars is proposed by the adherents of the light theory of Einstein, according to which a displacement of the spectral lines toward the red exists for all stars depending on the mass and the density of the stars by the formula

$$K = 0.634\ M^{2/3}\ \delta^{1/3}\ \text{km/sec.}$$

(cf. Sitter, *M. N.* 76 [1916]), where the mass M and the density δ of the star are measured in units of the sun. According to the results of Gyllenberg we have for the different spectral types

$$\begin{aligned}
B &: K = +0.91\ \text{Sir/st} = +3.84\ \text{km/sec} \\
A &: K = +0.03\ \text{Sir/st} = +0.13\ \text{km/sec} \\
F &: K = +0.05\ \text{Sir/st} = +0.21\ \text{km/sec} \\
G &: K = -0.17\ \text{Sir/st} = -0.72\ \text{km/sec} \\
K &: K = +0.75\ \text{Sir/st} = +3.16\ \text{km/sec} \\
M &: K = +1.11\ \text{Sir/st} = +4.68\ \text{km/sec}
\end{aligned}$$

The small density of the M stars (giants) makes difficult the explanation of the expansion by Einstein's formula. Even the B stars, for which the value of K seems to be well determined, give diverging results. Assuming, with Sitter, for these stars

$$M=10, \quad \delta=1/10$$

we get from the formula of Einstein

$$K=+1.4 \text{ km/sec}$$

which is only one-third of the observed value. We may therefore, with Sitter, conclude that "it is hardly a correct representation of the facts to say that Einstein's theory explains the observed systematic motion of the B stars." A similar conclusion was reached by Seelinger in *Astronomische Nachrichten* (*A. N.*) 4829 (1916).

(*d*) Another possible explanation is that the expansion is not directly dependent upon the spectral type but is to be considered as dependent upon the *distance*. Such a possibility indicates that the stars having the greatest expansion (B and probably also M stars) are most distant among the stars examined. In any case it is not impossible to decide whether the deciding factor is in the spectral type or in the distance (the parallax stars are, however, not sufficient for the object, the range of the distances is here too small). The questions were not, indeed, definitely solved in this manner, but that does not prevent seeking an explanation why the wavelength of a light vibration increases with the distance. It is to be observed that even the displacement of the spectral lines for very distant objects, the spiral nebulae, supports such an explanation.

For the present, however, we must consider this problem as one of the unsolved questions in stellar astronomy.

21. *Motion of the invariable plane.*—We have found in §6 that the galactic longitudes of the stars as well as their galactic latitudes upon an average are decreasing. The secular decrease of the latitude is evidently a direct consequence of the motion of the sun. The sun has a velocity component ($-W_\circ''$) amounting to +1.619 Siriometers per stellar year (or 1.619 sun-distances per year). The consequence is that the latitudes of the stars at the galactic equator must diminish. For calculating this decrease we may use the mean parallax of the stars brighter than the sixth magnitude, which according to §11 amounts to $0''.0127$. The stars must accordingly through the motion of the sun have an apparent displacement of the south perpendicular of the Galaxy amounting to $1.619\times 0''.0127=0''.0206$. The observed displacement is $0''.0150$. The difference between these values depends on the rather great value of v for the same squares at the galactic equator.

Regarding the secular decrease in the longitudes of the stars the explanation is different. From the decrease of the longitudes of $0''.0024$ per year we conclude that either the stellar system (the whole Galaxy) is turning indirectly through

this angle or that the point from which the longitudes are reckoned—i. e., the descending node of the equator on the Milky Way—is moving directly by this amount per year.

Eliminating the precession of the equator—the effect of which is shown in other ways—this motion of the equator would imply that the fundamental plane to which all motions in our planetary system ultimately may be reduced, *the invariable plane* of Laplace, would move in space so that its point of intersection with the Milky Way would proceed directly by 0″.0024 per year. Such a motion would be caused by the attraction of the stars on our planetary system. The interest of such a problem is that we would possibly obtain in such a manner a value of the total mass of the system of the Galaxy. Dr. Axel Jönsson, to whom I referred this problem, has found, however, that with any plausible assumption regarding the constitution of the Galaxy, no such explanation could be justified, the precession of the invariable plane being in all cases indirect and not direct as found from the observations.

It must therefore be concluded that the galactic plane (the Milky Way) is turning around the axis of the Galaxy with 0″.0024 per year in retrograde direction, giving a period of rotation of the Galaxy of 540 million years. This may be regarded as a rather slow rotation. Remembering, however, that the value 0″.0024 must be increased by the still undetermined precession of the invariable plane, we may expect a much shorter period of rotation of our Galaxy.

22. It may be considered singular that it should be possible to compute the rotation of the Galaxy without taking into regard objects outside the Milky Way (this rotation would be obtained most directly from the proper motions of the nebulae which are, however, unknown at present), but it is, indeed, no more singular than the possibility of determining with a gyroscopic compass the turnings of a ship. The planetary system is, indeed, just such a top allowing us, under certain assumptions, to determine whether it is the planetary system or the stars themselves that cause an observed change in their respective position.

CHAPTER III

MATHEMATICAL THEORY OF STELLAR VELOCITIES

A. Auxiliary Mathematical Theorems

23. *Theorems on determinants.*—Let $x_1, x_2, \ldots, x_n$ be n variables and consider the quadratic form

$$f = \Sigma a_{ij} x_i x_j$$

the summation being extended over both indices, i and j, from 1 to n.

Make a linear substitution

$$\begin{aligned} x_1 &= \gamma_{11}x_1' + \gamma_{12}x_2' + \ldots + \gamma_{1n}x_n', \\ x_2 &= \gamma_{21}x_1' + \gamma_{22}x_2' + \ldots + \gamma_{2n}x_n', \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ x_n &= \gamma_{n1}x_1' + \gamma_{n2}x_2' + \ldots + \gamma_{nn}x_n', \end{aligned}$$

then we have

$$f = \Sigma b_{ij} x_i' x_j'$$

and

(I) $$|b_{ij}| = |a_{ij}| D^2$$

where

$$D = |\gamma_{ij}| .$$

If the substitution is orthogonal we have $D^2 = 1$ and hence

(I*) $$|b_{ij}| = |a_{ij}| .$$

This theorem is common in all textbooks of determinants.

24. Put

$$A = |a_{ij}|$$

and let A_{ij} be the underdeterminant of A with respect to the element a_{ij}, so that

$$A_{ij} = \frac{\partial A}{\partial a_{ij}} .$$

Forming now a determinant B of these underdeterminants

$$B = |A_{ij}|$$

then we have

(II) $$B = A^{n-1}$$

which also is well known from the theory of determinants. If $n = 3$ we have

$$B = A^2$$

25. Let B_{ij} be an underdeterminant of B, so that

$$B_{ij} = \frac{\partial B}{\partial A_{ij}}$$

then we have

(III) $$\frac{a_{ij}}{A} = \frac{B_{ij}}{B} .$$

Even this theorem is to be found in textbooks (for instance that of Baltzer). The proof is, however, rather complicated. For $n=3$ it runs (B being equal to A^2)

$$B_{ij} = A a_{ij}$$

and is demonstrated (without determinants) by Lagrange in his *Pyramides triangulaires*. His proof is, as purely algebraical, rather circumstantial. The following short proof for any number of variables was given by Amanuensis Ohlsson.

Consider the elements in the ith row of the determinants $A = |a_{ij}|$. According to a fundamental theorem in the theory of determinants we then have

$$\begin{aligned}
&a_{i1}A_{11}+a_{i2}A_{12}+\ldots+a_{ij}A_{1j}+\ldots+a_{in}A_{1n}=0\\
&a_{i1}A_{21}+a_{i2}A_{22}+\ldots+a_{ij}A_{2j}+\ldots+a_{in}A_{2n}=0\\
&\text{— — — — — — — — — — — — —}\\
&a_{i1}A_{i1}+a_{i2}A_{i2}+\ldots+a_{ij}A_{ij}+\ldots+a_{in}A_{in}=A\\
&\text{— — — — — — — — — — — — —}\\
&a_{i1}A_{n1}+a_{i2}A_{n2}+\ldots+a_{ij}A_{nj}+\ldots+a_{in}A_{nn}=0.
\end{aligned}$$

Solving these n equations regarding $a_{i1}, a_{i2}, \ldots, a_{in}$, we obtain

$$a_{ij}B = AB_{ij}$$

which is the theorem sought.

26. *Theorem of Wicksell.*—Let us define a group of quantities c_{ij} through the formulae

(IVa) $$c_{ij} = \frac{1}{A}\frac{\partial A}{\partial a_{ij}} = \frac{A_{ij}}{A}$$

and form the determinant

$$C = |c_{ij}|$$

we then have:

(IVb) $$a_{ij} = \frac{1}{C}\frac{\partial C}{\partial c_{ij}} = \frac{C_{ij}}{C}$$

and

(IVc) $$AC = 1$$

For proving IVc we first observe that

$$C = |c_{ij}| = \left|\frac{A_{ij}}{A}\right| = \frac{1}{A^n}\cdot\left|A_{ij}\right| = \frac{B}{A^n} = \frac{A^{n-1}}{A^n} = \frac{1}{A} .$$

so that

$$AC = 1$$

which is the theorem (IVc).

Let us now consider the determinant

$$C = \left| \frac{A_{ij}}{A} \right|$$

and form the subdeterminant C_{ij}. This being obtained by putting 1 for $A_{ij} : A$ and 0 for all other elements in the ith row and the jth column, we evidently have

$$C_{ij} = \frac{B_{ij}}{A^{n-1}} = \frac{B_{ij}}{B}$$

or according to (III)

$$C_{ij} = \frac{a_{ij}}{A}$$

or according to (IVc)

$$C_{ij} = Ca_{ij}$$

so that

$$a_{ij} = C_{ij}/C$$

which is the theorem (IVb).

This important theorem has been given by Wicksell in *L. M.* 95.

27. *Theorem of Jacobi.*—Let

$$\begin{aligned} 1 = {} & a_{11}x^2 + a_{12}xy + a_{13}xz \\ & + a_{21}yx + a_{22}y^2 + a_{23}yz \\ & + a_{31}zx + a_{32}zy + a_{33}z^2 \end{aligned}$$

be the equation of an ellipsoid, and let σ_1, σ_2, and σ_3 be the three semiaxes of this ellipsoid. Then the values of

$$1/\sigma_1^2, \quad 1/\sigma_2^2, \quad 1/\sigma_3^2$$

are equal to the roots of the equation

$$I(s) = \begin{vmatrix} a_{11}-s, & a_{12}, & a_{13} \\ a_{21}, & a_{22}-s, & a_{23} \\ a_{31}, & a_{32}, & a_{33}-s \end{vmatrix} = 0$$

which is the equation of Jacobi. Each of the three roots of this equation determines the position of one of the principal axes. Let s denote any of the roots and let l, m, n be the direction cosines of the corresponding principal axis then we have

$$\frac{l}{I_{11}} = \frac{m}{I_{12}} = \frac{n}{I_{13}} = \frac{1}{\sqrt{I_{11}^2 + I_{12}^2 + I_{13}^2}}$$

where I_{ij} denotes the subdeterminant of the determinant $I(s)$.

The equation $I(s) = 0$ may also be written

$$\begin{aligned} 0 = {} & s^3 - s^2(a_{11}+a_{22}+a_{33}) \\ & + s(a_{22}a_{33} + a_{33}a_{11} + a_{11}a_{22} - a_{23}^2 - a_{31}^2 - a_{12}^2) \\ & - a_{11}a_{22}a_{33} - 2a_{23}a_{31}a_{12} + a_{11}a_{23}^2 + a_{22}a_{31}^2 + a_{33}a_{12}^2. \end{aligned}$$

These propositions are well known in analytical geometry.

28. *Reciprocal ellipsoids.*—Let E_1 and E_2 be two ellipsoids defined through the equations;

$$E_1: \quad 1 = a_{11}x^2 + a_{12}xy + a_{13}xz + a_{21}yx + a_{22}y^2 + a_{23}yz + a_{31}zx + a_{32}zy + a_{33}z^2$$

and

$$E_2: \quad 1 = c_{11}x^2 + c_{12}xy + c_{13}xz + c_{21}yx + c_{22}y^2 + c_{23}yz + c_{31}zx + c_{32}zy + c_{33}z^2$$

and suppose that between the coefficients a_{ij} and c_{ij} exist the relations

$$a_{ij} = \frac{1}{C}\frac{\partial C}{\partial c_{ij}},$$

$$c_{ij} = \frac{1}{A}\frac{\partial A}{\partial a_{ij}}$$

the latter relation being, according to (IVa), a direct conclusion from the first.

Then the ellipsoids E_1 and E_2 are called *reciprocal* ellipsoids, the semiaxes of the first ellipsoid being equal to the inverse value of the semiaxes of the other.

The semiaxes of E_1 are, indeed, determined by the roots of the equation

$$0 = F(s) = \begin{vmatrix} a_{11}-s, & a_{12}, & a_{13} \\ a_{21}, & a_{22}-s, & a_{23} \\ a_{31}, & a_{32}, & a_{33}-s \end{vmatrix}$$

and those of E_2 by

$$0 = G(t) = \begin{vmatrix} c_{11}-t, & c_{12}, & c_{13} \\ c_{21}, & c_{22}-t, & c_{23} \\ c_{31}, & c_{32}, & c_{33}-t \end{vmatrix}$$

Developing $F(s)$ into powers of s we have

$$F(s) = A + \left(\frac{\partial F}{\partial s}\right)_{s=0} s + \frac{1}{2}\left(\frac{\partial^2 F}{\partial s^2}\right)_{s=0} s^2 - s^3$$

But

$$\frac{\partial F}{\partial s} = -\left(\frac{\partial F}{\partial a_{11}} + \frac{\partial F}{\partial a_{22}} + \frac{\partial F}{\partial a_{33}}\right)$$

$$\frac{\partial^2 F}{\partial s^2} = \frac{\partial^2 F}{\partial a_{11}\partial a_{22}} + \frac{\partial^2 F}{\partial a_{11}\partial a_{33}} + \frac{\partial^2 F}{\partial a_{22}\partial a_{33}} + \frac{\partial^2 F}{\partial a_{22}\partial a_{11}} + \frac{\partial^2 F}{\partial a_{33}\partial a_{11}} + \frac{\partial^2 F}{\partial a_{33}\partial a_{22}}$$

$$= 2\left(\frac{\partial^2 F}{\partial a_{22}\partial a_{33}} + \frac{\partial^2 F}{\partial a_{33}\partial a_{11}} + \frac{\partial^2 F}{\partial a_{11}\partial a_{22}}\right)$$

$$= 2\,(a_{11}-s+a_{22}-s+a_{33}-s)$$

and hence for $s=0$

$$\left(\frac{\partial F}{\partial s}\right)_{s=0} = -\left(\frac{\partial A}{\partial a_{11}} + \frac{\partial A}{\partial a_{22}} + \frac{\partial A}{\partial a_{33}}\right)$$

$$= -(A_{11}+A_{22}+A_{33})$$

$$\left(\frac{\partial^2 F}{\partial s^2}\right)_{s=0} = 2\,(a_{11}+a_{22}+a_{33})$$

so that

$$F(s) = A - s(A_{11}+A_{22}+A_{33}) + s^2(a_{11}+a_{22}+a_{33}) - s^3$$

and in like manner

$$G(t) = C - t(C_{11}+C_{22}+C_{33}) + t^2(c_{11}+c_{22}+c_{33}) - t^3$$

Substituting, however, in the former, according to (IV)

$$\begin{aligned} A &= 1/C \\ A_{11} &= Ac_{11} = c_{11}/C \\ A_{22} &= Ac_{22} = c_{22}/C \\ A_{33} &= Ac_{33} = c_{33}/C \\ a_{11} &= C_{11}/C, \\ a_{22} &= C_{22}/C \\ a_{33} &= C_{33}/C \end{aligned}$$

we get

$$\begin{aligned} F(s) &= \frac{1}{C} - s\left(\frac{c_{11}}{C}+\frac{c_{22}}{C}+\frac{c_{33}}{C}\right) + s^2\left(\frac{C_{11}}{C}+\frac{C_{22}}{C}+\frac{C_{33}}{C}\right) - s^3 \\ &= \frac{s^3}{C}\left[\frac{1}{s^3} - \frac{1}{s^2}(c_{11}+c_{22}+c_{33}) + \frac{1}{s}(C_{11}+C_{22}+C_{33}) - C\right] \end{aligned}$$

and comparing with $G(t) = 0$, we find

$$s = 1/t,$$

so that the value of the roots of the two equations, and hence also of the axes, are inversed.

This theorem has also been given by Wicksell (*loc. cit.*).

It does not follow from the foregoing demonstration that even the directions of the axes in the two reciprocal ellipsoids coincide. The direction cosines l, m, n of one principal axis in E_1 are given by

$$\frac{l}{F_{11}} = \frac{m}{F_{12}} = \frac{n}{F_{13}}$$

and the direction cosines l', m', n', of the corresponding axis in E_2 through

$$\frac{l'}{G_{11}} = \frac{m'}{G_{12}} = \frac{n'}{G_{13}}$$

where we have put $t = 1/s$.

We have, however, as is easily found

$$\begin{aligned} F_{11} &= \frac{\partial F}{\partial a_{11}} = A_{11} - s(a_{22}+a_{33}) + s^2 \\ G_{11} &= \frac{\partial G}{\partial c_{11}} = C_{11} - t(c_{22}+c_{33}) + t^2 \end{aligned}$$

Observing that t is a root of the equation $G(t)=0$, so that, according to our development of $G(t)$ before given, we have

$$0=C-t(C_{11}+C_{22}+C_{33})+t^2(c_{11}+c_{22}+c_{33})-t^3$$

or even

$$0=\frac{C}{t}-(C_{11}+C_{22}+C_{33})+t(c_{11}+c_{22}+c_{33})-t^2$$

and adding the right member in this equation to the expression for G_{11} we get

$$\begin{aligned}
G_{11}&=\frac{C}{t}-(C_{11}+C_{22}+C_{33})+t(c_{11}+c_{22}+c_{33})-t^2+C_{11}-t(c_{22}+c_{33})+t^2\\
&=\frac{C}{t}-(C_{22}+C_{33})+tc_{11}\\
&=Cs-(C_{22}+C_{33})+\frac{c_{11}}{s}\\
&=Cs-C(a_{22}+a_{33})+\frac{c_{11}}{s}\\
&=C\left[s-(a_{22}+a_{33})+\frac{A_{11}}{s}\right]\\
&=\frac{C}{s}\left[s^2-(a_{22}+a_{33})s+A_{11}\right]\\
&=\frac{C}{s}F_{11}
\end{aligned}$$

using the relation between c_{ij}, C_{ij}, a_{ij}, A_{ij}.

Further we get

$$\begin{aligned}
F_{12}&=a_{23}a_{31}-a_{21}a_{33}+sa_{21}\\
&=A_{12}+sa_{21}=A_{12}+sa_{12}\\
G_{12}&=C_{12}+tc_{21}=C_{12}+tc_{12}\\
&=Ca_{12}+tA_{12}C=C(a_{12}+tA_{12})\\
&=Ct(a_{12}s+A_{12})=\frac{C}{s}(A_{12}+sa_{12})\\
&=\frac{C}{s}F_{12}
\end{aligned}$$

and

$$\begin{aligned}
F_{13}&=a_{21}a_{32}-a_{31}a_{22}+a_{31}s\\
&=A_{13}+a_{31}s=A_{13}+a_{13}s\\
G_{13}&=C_{13}+c_{13}t\\
&=Ca_{13}+tA_{13}C=C(a_{13}+tA_{13})\\
&=Ct\left(\frac{a_{13}}{t}+A_{13}\right)=\frac{C}{s}(A_{13}+a_{13}s)\\
&=\frac{C}{s}F_{13}.
\end{aligned}$$

From the relations

$$\frac{G_{11}}{F_{11}}=\frac{G_{12}}{F_{12}}=\frac{G_{13}}{F_{13}}=\frac{C}{s}$$

we conclude that the direction cosines l, m, n are identical with l', m', n'. The directions of the axes in the reciprocal ellipsoids coincide.

29. *Moments.*—Let U, V, W denote the velocity components of a star in any system of coördinates. The given population of stars may consist of N individuals for each of which these components are known. To determine the characteristics in the frequency distribution of these components we have first to compute the moments of the frequency function. Let $F(U, V, W)$ be the frequency function so that

$$F(U, V, W)\, dU\, dV\, dW$$

is the number of stars having a velocity between the limits

$$U \pm \frac{1}{2} dU,\ V \pm \frac{1}{2} dV,\ W \pm \frac{1}{2} dW.$$

The moment μ'_{ijk} about any "point" (U_1, V_1, W_1) is defined through

$$\mu'_{ijk} = \iiint (U-U_1)^i (V-V_1)^j (W-W_1)^k F\, dU\, dV\, dW \tag{1}$$

the integrals being taken over all values of U, V, and W between $-\infty$ to $+\infty$.

Especially we have

$$\mu'_{000} = \iiint F\, dU dV dW = N. \tag{2}$$

Besides the moments (the "absolute" moments) μ'_{ijk} we consider the *relative* moments ν'_{ijk}

$$\nu'_{ijk} = \mu'_{ijk} : \mu'_{000} = \mu'_{ijk} : N. \tag{3}$$

The *mean values* $\overline{U}$, $\overline{V}$, $\overline{W}$ of the components U, V, W are defined by the formulae;

$$\begin{aligned} \overline{U} &= \iiint U\, F\, dU dV dW : N = \nu'_{100} + U_1 \\ \overline{V} &= \nu'_{010} + V_1 \\ \overline{W} &= \nu'_{001} + W_1 \end{aligned} \tag{4}$$

Moments taken about the means $\overline{U}$, $\overline{V}$, $\overline{W}$ are denoted by μ_{ijk} and ν_{ijk}, so that

$$\begin{aligned} \mu_{ijk} &= \iiint (U-\overline{U})^i (V-\overline{V})^j (W-\overline{W})^k F\, dU dV dW \\ \nu_{ijk} &= \mu_{ijk} : N \end{aligned} \tag{5}$$

From the definition (4) of $\overline{U}$, $\overline{V}$, $\overline{W}$ it follows that

$$\nu_{100} = \nu_{010} = \nu_{001} = 0 \tag{6}$$

30. *Relations between ν'_{ijk} and ν_{ijk}.*—In numerical calculations we first compute the moments ν'_{ijk} about a given point U_1, V_1, W_1, whereas for the following considerations we may use the moments ν_{ijk} about the means. It is therefore necessary to formulate the relations between these moments.

Putting

$$\begin{aligned} \overline{U}-U_1&=a\\ \overline{V}-V_1&=b\\ \overline{W}-W_1&=c \end{aligned} \tag{7}$$

we have

$$\nu'_{ijk}=\iiint (U-\overline{U}+a)^i(V-\overline{V}+b)^j(W-\overline{W}+c)^k F\,dUdVdW : N$$

$$\nu_{ijk}=\iiint (U-U_1-a)^i(V-V_1-b)^j(W-W_1-c)^k F\,dUdVdW : N$$

and developing the polynomials in the integrals into powers of a, b, and c we get directly the relations sought. For all moments to the fourth order we obtain the following system of formulae:

$$\begin{aligned} \nu'_{100}&=\overline{U}-U_1=a & \nu_{100}&=0\\ \nu'_{010}&=\overline{V}-V_1=b & \nu_{010}&=0\\ \nu'_{001}&=\overline{W}-W_1=c & \nu_{001}&=0 \end{aligned}$$

$$\begin{aligned} \nu'_{200}&=\nu_{200}+a^2; & \nu_{200}&=\nu'_{200}-a^2\\ \nu'_{020}&=\nu_{020}+b^2; & \nu_{020}&=\nu'_{020}-b^2\\ \nu'_{002}&=\nu_{002}+c^2; & \nu_{002}&=\nu'_{002}-c^2\\ \nu'_{011}&=\nu_{011}+bc; & \nu_{011}&=\nu'_{011}-bc\\ \nu'_{101}&=\nu_{101}+ca; & \nu_{101}&=\nu'_{101}-ca\\ \nu'_{110}&=\nu_{110}+ab; & \nu_{110}&=\nu'_{110}-ab \end{aligned}$$

$$\begin{aligned} \nu'_{300}&=\nu_{300}+3a\nu_{200}+a^3\\ \nu'_{030}&=\nu_{030}+3b\nu_{020}+b^3\\ \nu'_{003}&=\nu_{003}+3c\nu_{002}+c^3\\ \nu'_{021}&=\nu_{021}+c\nu_{020}+2b\nu_{011}+b^2c\\ \nu'_{012}&=\nu_{012}+b\nu_{002}+2c\nu_{011}+bc^2\\ \nu'_{201}&=\nu_{201}+c\nu_{200}+2a\nu_{101}+a^2c\\ \nu'_{102}&=\nu_{102}+a\nu_{002}+2c\nu_{101}+ac^2\\ \nu'_{210}&=\nu_{210}+b\nu_{200}+2a\nu_{110}+a^2b\\ \nu'_{120}&=\nu_{120}+a\nu_{020}+2b\nu_{110}+ab^2\\ \nu'_{111}&=\nu_{111}+a\nu_{011}+b\nu_{101}+c\nu_{110}+abc \end{aligned}$$

$$\begin{aligned} \nu_{300}&=\nu'_{300}-3a\nu'_{200}+2a^3\\ \nu_{030}&=\nu'_{030}-3b\nu'_{020}+2b^3\\ \nu_{003}&=\nu'_{003}-3c\nu'_{002}+2c^3\\ \nu_{021}&=\nu'_{021}-c\nu'_{020}-2b\nu'_{011}+2b^2c\\ \nu_{012}&=\nu'_{012}-b\nu'_{002}-2c\nu'_{011}+2bc^2\\ \nu_{201}&=\nu'_{201}-c\nu'_{200}-2a\nu'_{101}+2a^2c\\ \nu_{102}&=\nu'_{102}-a\nu'_{002}-2c\nu'_{101}+2ac^2\\ \nu_{210}&=\nu'_{210}-b\nu'_{200}-2a\nu'_{110}+2a^2b\\ \nu_{120}&=\nu'_{120}-a\nu'_{020}-2b\nu'_{110}+2ab^2\\ \nu_{111}&=\nu'_{111}-a\nu'_{011}-b\nu'_{101}-c\nu'_{110}+2abc \end{aligned}$$

$$
\begin{aligned}
\nu'_{400} &= \nu_{400}+4a\nu_{300}+6a^2\nu_{200}+a^4\\
\nu'_{040} &= \nu_{040}+4b\nu_{030}+6b^2\nu_{020}+b^4\\
\nu'_{004} &= \nu_{004}+4c\nu_{003}+6c^2\nu_{002}+c^4\\
\nu'_{031} &= \nu_{031}+c\nu_{030}+3b\nu_{021}+3bc\nu_{020}+3b^2\nu_{011}+b^3c\\
\nu'_{013} &= \nu_{013}+b\nu_{003}+3c\nu_{012}+3bc\nu_{002}+3c^2\nu_{011}+bc^3\\
\nu'_{301} &= \nu_{301}+c\nu_{300}+3a\nu_{201}+3ac\nu_{200}+3a^2\nu_{101}+a^3c\\
\nu'_{103} &= \nu_{103}+a\nu_{003}+3c\nu_{102}+3ac\nu_{002}+3c^2\nu_{101}+ac^3\\
\nu'_{310} &= \nu_{310}+b\nu_{300}+3a\nu_{210}+3ab\nu_{200}+3a^2\nu_{110}+a^3b\\
\nu'_{130} &= \nu_{130}+a\nu_{030}+3b\nu_{120}+3ab\nu_{020}+3b^2\nu_{110}+ab^3\\
\nu'_{022} &= \nu_{022}+2c\nu_{021}+2b\nu_{012}+c^2\nu_{020}+b^2\nu_{002}+4bc\nu_{011}+b^2c^2\\
\nu'_{202} &= \nu_{202}+2c\nu_{201}+2a\nu_{102}+c^2\nu_{200}+a^2\nu_{002}+4ac\nu_{101}+a^2c^2\\
\nu'_{220} &= \nu_{220}+2b\nu_{210}+2a\nu_{120}+b^2\nu_{200}+a^2\nu_{020}+4ab\nu_{110}+a^2b^2
\end{aligned}
$$

$$
\begin{aligned}
\nu_{400} &= \nu'_{400}-4a\nu'_{300}+6a^2\nu'_{200}-3a^4\\
\nu_{040} &= \nu'_{040}-4b\nu'_{030}+6b^2\nu'_{020}-3b^4\\
\nu_{004} &= \nu'_{004}-4c\nu_{003}+6c^2\nu'_{002}-3c^4\\
\nu_{031} &= \nu'_{031}-c\nu'_{030}-3b\nu'_{021}+3bc\nu'_{020}+3b^2\nu'_{011}-3b^3c\\
\nu_{013} &= \nu'_{013}-b\nu'_{003}-3c\nu'_{012}+3bc\nu'_{002}+3c^2\nu'_{011}-3bc^3\\
\nu_{301} &= \nu'_{301}-c\nu'_{300}-3a\nu'_{201}+3ac\nu'_{200}+3a^2\nu'_{101}-3a^3c\\
\nu_{103} &= \nu'_{103}-a\nu'_{003}-3c\nu'_{102}+3ac\nu'_{002}+3c^2\nu'_{101}-3ac^3\\
\nu_{310} &= \nu'_{310}-b\nu'_{300}-3a\nu'_{210}+3ab\nu'_{200}+3a^2\nu'_{110}-3a^3b\\
\nu_{130} &= \nu'_{130}-a\nu'_{030}-3b\nu'_{120}+3ab\nu'_{020}+3b^2\nu'_{110}-3ab^3\\
\nu_{022} &= \nu'_{022}-2c\nu'_{021}-2b\nu'_{012}+c^2\nu'_{020}+b^2\nu'_{002}+4bc\nu'_{011}-3b^2c^2\\
\nu_{202} &= \nu'_{202}-2c\nu'_{201}-2a\nu'_{102}+c^2\nu'_{200}+a^2\nu'_{002}+4ac\nu'_{101}-3a^2c^2\\
\nu_{220} &= \nu'_{220}-2b\nu'_{210}-2a\nu'_{120}+b^2\nu'_{200}+a^2\nu'_{020}+4ab\nu'_{110}-3a^2b^2
\end{aligned}
$$

In the numerical calculation of the moments, the moments μ'_{ijk} are first computed. Since calculation would be too tedious if the exact velocity of each individual star were taken into consideration, the population of stars is first distributed into subclasses of two dimensions (YZ, ZX, XY) as is done in a correlation table. The only moment, lower than the fifth order, for which an individual computation is necessary (though even here a division into classes can be made), is μ'_{111}. The computation of μ'_{ijk} being finished the values of ν'_{ijk} are obtained by dividing through N and then ν_{ijk} using the formulae given in this paragraph.

B. Characteristics of the Second Order

31. *The velocity ellipsoid.*—The frequency function F of the velocities of the stars being of type A, we have the following analytical form of F.

We put

$$
\begin{aligned}
x &= U-\overline{U}\\
y &= V-\overline{V}\\
z &= W-\overline{W}
\end{aligned}
\tag{8}
$$

and have then (dividing by N so that F is the *relative* frequency function)

$$F(x, y, z) = \varphi(x, y, z) + \sum_{i+j+k=3}^{\infty} A_{ijk} \frac{\partial^{i+j+k}\varphi}{\partial x^i \partial y^j \partial z^k} \tag{9}$$

where the *generating function* $\varphi(x,y,z)$ has the form

$$\varphi(x,y,z) = H\, e^{-\frac{f}{2}} \tag{10}$$

and

$$\begin{aligned} f = {} & a_{200}x^2 + a_{110}xy + a_{101}xz \\ & + a_{110}yx + a_{020}y^2 + a_{011}yz \\ & + a_{101}zx + a_{011}zy + a_{002}z^2 \\ = {} & \sum a_{ijk} x^i y^j z^k \end{aligned} \tag{11}$$

The characteristics a_{ijk}, A_{ijk}, and H may all be expressed through the moments ν_{ijk} of F. We will show how this determination is performed, working out in this chapter the calculation for the characteristics of the second order (a_{ijk} and also H).

For determining H we observe that the function F being the *relative* frequency function we have

$$\int_{-\infty}^{+\infty}\int\int F\, dxdydz = 1$$

and hence also

$$\iiint \varphi\, dxdydz = H \iiint e^{-\frac{f}{2}} dxdydz = 1 \tag{12}$$

For performing this integration we turn the system of coördinates putting

$$\begin{aligned} x &= \epsilon_{11}x' + \epsilon_{12}y' + \epsilon_{13}z' \\ y &= \epsilon_{21}x' + \epsilon_{22}y' + \epsilon_{23}z' \\ z &= \epsilon_{31}x' + \epsilon_{32}y' + \epsilon_{33}z' \end{aligned}$$

where

$$|\epsilon_{ij}| = +1.$$

We then obtain

$$dxdydz = dx'dy'dz'.$$

The function f now has the form

$$f = \sum c_{ijk} x'^i y'^j z'^k$$

and, according to §23, we have

$$A = |a_{ijk}| = |c_{ijk}| = C. \tag{13}$$

We determine the new system of coördinates in such a manner that

$$\begin{aligned} f &= c_{200}x'^2 + c_{020}y'^2 + c_{002}z'^2 \\ &= \frac{x'^2}{\sigma_1^2} + \frac{y'^2}{\sigma_2^2} + \frac{z'^2}{\sigma_3^2} \end{aligned}$$

Then we get from (12)

$$1 = H \int_{-\infty}^{+\infty} \iiint e^{-\frac{x'^2}{2\sigma_1^2} - \frac{y'^2}{2\sigma_2^2} - \frac{z'^2}{2\sigma_3^2}} dx'dy'dz'$$

The variables being here separated, the integrations may be performed directly, according to the known formula of Laplace, and we get

$$1 = H\sigma_1\sigma_2\sigma_3(2\pi)^{3/2}$$

or

$$H = \frac{1}{\sigma_1\sigma_2\sigma_3(2\pi)^{3/2}}.$$

However,

$$\sigma_1\sigma_2\sigma_3 = 1 : \sqrt{c_{200}c_{020}c_{002}}$$
$$= 1 : \sqrt{C} = 1 : \sqrt{A}$$

and we get

(14) $$H = \frac{\sqrt{A}}{(2\pi)^{3/2}}$$

and

$$\varphi(x, y, z) = \frac{\sqrt{A}}{(2\pi)^{3/2}} e^{-\frac{1}{2}\sum a_{ijk}x^i y^j z^k}$$

and, according to (12),

$$1 = \frac{\sqrt{A}}{(2\pi)^{3/2}} \iiint dxdydz\, e^{-\frac{1}{2}\sum a_{ijk}x^i y^j z^k}$$

which equation also may be written

$$\frac{1}{\sqrt{A}} = \frac{1}{(2\pi)^{3/2}} \iiint dxdydz\, e^{-\frac{1}{2}\sum a_{ijk}x^i y^j z^k}$$
$$(i+j+k=2).$$

We derive both sides of this equation relative to any one of the coefficients a_{ijk} and obtain

$$\frac{1}{A^{3/2}} \frac{\partial A}{\partial a_{ijk}} = \frac{1}{(2\pi)^{3/2}} \iiint x^i y^j z^k dxdydz\, e^{-\frac{1}{2}\sum a_{ijk}x^i y^j z^k}$$

or

$$\frac{1}{A} \frac{\partial A}{\partial a_{ijk}} = \frac{\sqrt{A}}{(2\pi)^{3/2}} \iiint x^i y^j z^k dxdydz\, e^{-\frac{1}{2}\sum a_{ijk}x^i y^j z^k}$$

(15) $$= \iiint x^i y^j z^k dxdydz\, \varphi(x, y, z).$$

When making derivations relative to a_{ijk}, we may observe that the coefficients a_{ijk}, excepting the instance where one of the indices i, j, k is equal to 2, occur at two different places in the form f as well as in the determinant A. It is supposed that the derivation takes place only regarding one of these values (otherwise we would have to divide by 2).

The right member of (15) is equal to a moment of the auxiliary function $\varphi(x, y, z)$, which we may call λ_{ijk}. In all the formulae above we have

$$i+j+k \leqq 2$$

and it follows then directly from (9) that the moments of φ coincide with the corresponding moments ν_{ijk} of F so that

$$\nu_{ijk}=\lambda_{ijk} \qquad (i+j+k \leqq 2).$$

Our equation (15) now takes the form

$$\nu_{ijk}=\frac{1}{A}\frac{\partial A}{\partial a_{ijk}}=\frac{1}{A}A_{ijk} \tag{16}$$

where A_{ijk} is the subdeterminant of A regarding the element a_{ijk}.

We may now use the determinant theorem of Wicksell in §26. For this purpose we have to form the determinant M of the moments ν_{ijk}

$$M=|\nu_{ijk}|$$

and the inversion of (16) is then

$$a_{ijk}=\frac{1}{M}\frac{\partial M}{\partial \nu_{ijk}}=\frac{M_{ijk}}{M}. \tag{17}$$

Through this formula the characteristics a_{ijk} are expressed by the moments ν_{ijk} of the function F.

32. *The velocity ellipsoid.*—The surface defined through the equation

$$\begin{aligned} f=&a_{200}x^2+a_{020}y^2+a_{002}z^2 \\ &+2a_{011}yz+2a_{101}zx+2a_{110}xy=\text{constant} \end{aligned} \tag{18}$$

is called the velocity ellipsoid. All velocity components of the stars satisfying this equation have the same frequency, as far as characteristics of the third and higher order are neglected.

Turning the system of the coördinates as in the preceding article, this equation takes the form (taking the constant $=1$)

$$\frac{x'^2}{\sigma_1^2}+\frac{y'^2}{\sigma_2^2}+\frac{z'^2}{\sigma_3^2}=1. \tag{19}$$

With the help of the velocity ellipsoid we may compute the number of stars moving in any direction. Introducing polar coördinates we put

$$\begin{aligned} x'&=\omega\cos\theta\cos\psi \\ y'&=\omega\cos\theta\sin\psi \\ z'&=\omega\sin\theta \end{aligned} \tag{20}$$

so that $\omega=$the total velocity in the direction (θ, ψ).

Now the number of stars having velocity components between the limits

$$x' \pm \frac{1}{2} dx'$$
$$y' \pm \frac{1}{2} dy'$$
$$z' \pm \frac{1}{2} dz'$$

is

$$F\, dx'dy'dz' = \frac{dx'dy'dz'}{\sigma_1\sigma_2\sigma_3(2\pi)^{3/2}} \cdot e^{-\frac{x'^2}{2\sigma_1^2} - \frac{y'^2}{2\sigma_2^2} - \frac{z'^2}{2\sigma_3^2}} \tag{21}$$

we obtain for the number of stars having a total velocity between $\omega - \frac{1}{2} d\omega$ and $\omega + \frac{1}{2} d\omega$ in a direction determined by the limits

$$\theta \pm \frac{1}{2} d\theta \tag{22}$$
$$\psi \pm \frac{1}{2} d\psi$$

the expression

$$F\, \omega^2 \cos\theta d\omega d\theta d\psi = \frac{\omega^2 \cos\theta d\omega d\theta d\psi}{\sigma_1\sigma_2\sigma_3(2\pi)^{3/2}}\ e^{-\frac{\omega^2}{2}\left(\frac{\cos^2\theta\cos^2\psi}{\sigma_1^2} + \frac{\cos^2\theta\sin^2\psi}{\sigma_2^2} + \frac{\sin^2\theta}{\sigma_3^2}\right)} \tag{23}$$

If, however, the radius vector in the velocity ellipsoid (19) is denoted by ρ, this expression (23) may be written

$$\frac{1}{\sigma_1\sigma_2\sigma_3(2\pi)^{3/2}}\, e^{-\frac{\omega^2}{2\rho^2}}\ \omega^2 \cos\theta d\omega d\theta d\psi$$

where

$$\frac{1}{\rho^2} = \frac{\cos^2\theta\cos^2\psi}{\sigma_1^2} + \frac{\cos^2\theta\sin^2\psi}{\sigma_2^2} + \frac{\sin^2\theta}{\sigma_3^2}.$$

For obtaining the number of stars moving in a certain direction we keep θ and ψ constant, in which case even ρ is unaltered, and integrate the expression above over all values of ω between $\omega = 0$ and $\omega = +\infty$. We then get

$$\frac{\cos\theta d\theta d\psi}{\sigma_1\sigma_2\sigma_3(2\pi)^{3/2}} \int_0^\infty e^{-\frac{\omega^2}{2\rho^2}} \cdot \omega^2 d\omega$$

which may be directly integrated and has the value

$$\frac{\cos\theta d\theta d\psi}{4\pi \cdot \sigma_1\sigma_2\sigma_3} \cdot \rho^3 \tag{24}$$

which is the relative number of stars moving in the direction (22) or more exactly in the direction defined through the limits (22). We may observe that

$$\frac{4}{3}\pi\sigma_1\sigma_2\sigma_3$$

is the volume of the velocity ellipsoid and that

$$\frac{1}{3}\rho^3 \cos\theta . d\theta . d\psi$$

is the volume of a cone having its top in the center of the ellipsoid, of height ρ and angle of aperture $d\psi . \cos\theta d\theta$.

It follows from (24) that the number of stars in the direction (22) *is proportional to the third power of the radius vector of the velocity ellipsoid* in the same direction. The number of stars moving in the direction of the three axes of the ellipsoid are hence in the proportion

$$\sigma_1^3 : \sigma_2^3 : \sigma_3^3 : \quad .$$

33. *Velocity distribution in two dimensions.*—Integrating (21) over all values of z between $-\infty$ and $+\infty$ we obtain the number of stars having velocity components between the limits

$$x' \pm \frac{1}{2}dx'$$

$$y' \pm \frac{1}{2}dy'$$

$$F\,dx'dy' = \frac{dx'dy'}{\sigma_1\sigma_2 2\pi}\, e^{-\frac{1}{2}\left(\frac{x'^2}{\sigma_1^2}+\frac{y'^2}{\sigma_2^2}\right)}$$

Introducing polar coördinates

$$x' = \omega\cos\psi, \quad y' = \omega\sin\psi$$

we obtain for the number of stars with a total velocity in the plane $(x'y')$ between $\omega - \frac{1}{2}d\omega$ and $\omega + \frac{1}{2}d\omega$ and moving in a direction between the limits

$$\psi \pm \frac{1}{2}d\psi$$

the expression

$$F dx'dy' = \frac{\omega d\omega d\psi}{\sigma_1\sigma_2 2\pi}\, e^{-\frac{\omega^2}{2}\left(\frac{\cos^2\psi}{\sigma_1^2}+\frac{\sin^2\psi}{\sigma_2^2}\right)} \tag{25}$$

The curve defined through the equation

$$\frac{x'^2}{\sigma_1^2}+\frac{y'^2}{\sigma_2^2}=1$$

may be called the velocity ellipse in the plane $(x'y')$ and the radius vector ρ of this ellipse in the direction ψ has the expression

$$\frac{1}{\rho^2}=\frac{\cos^2\psi}{\sigma_1^2}+\frac{\sin^2\psi}{\sigma_2^2}$$

so that (25) may be written

$$F\,dx'dy' = \frac{\omega d\omega d\psi}{\sigma_1\sigma_2 2\pi}\, e^{-\frac{\omega^2}{2\rho^2}}$$

Keeping here ψ unaltered and integrating over all values of ω we obtain for the number of stars moving in the direction $\psi \pm \frac{1}{2}d\psi$ the expression

$$\frac{d\psi}{2\pi\sigma_1\sigma_2}\,\rho^2$$

Hence the number here is proportional to the square of the radius vector of the velocity ellipse in the direction ψ.

34. *The moment ellipsoid.*—We have found in §31 the following relations between the characteristics a_{ijk} and the moments ν_{ijk}

$$a_{ijk} = \frac{1}{M}\frac{\partial M}{\partial \nu_{ijk}}$$

$$\nu_{ijk} = \frac{1}{A}\frac{\partial A}{\partial a_{ijk}}$$

where

$$M = |\,\nu_{ijk}\,|$$
$$A = |\,a_{ijk}\,|\,.$$

According to the theorem of Wicksell in §28 we conclude that the two surfaces

(a) $$\sum a_{ijk}x^i y^j z^k = 1 \qquad (i+j+k=2)$$

and

(m) $$\sum \nu_{ijk}x^i y^j z^k = 1 \qquad (i+j+k=2)$$

represent two reciprocal ellipsoids, the axes in both being equally directed but having inverse values. The surface (m) may be called the moment ellipsoid.

It follows from this relation that, for the study of the velocity ellipsoid, it is not at all necessary to compute the values of a_{ijk}. The values of the squares of the axes σ_1, σ_2, σ_3 are, indeed, directly found as roots of the equation

(26) $$G(t) = \begin{vmatrix} \nu_{200}-t, & \nu_{110} & \nu_{101} \\ \nu_{110}, & \nu_{020}-t, & \nu_{011} \\ \nu_{101}, & \nu_{011}, & \nu_{002}-t \end{vmatrix} = 0$$

and the direction cosines l, m, n of these axes are obtained from

(27) $$\frac{l}{G_{11}} = \frac{m}{G_{12}} = \frac{n}{G_{13}} = \frac{1}{\sqrt{G^2_{11}+G^2_{12}+G^2_{13}}}\,.$$

The moment ellipsoid may also be used to another purpose. Let us determine the dispersion in the components of the velocity in any direction (θ, ψ), the angles being referred to the principal axes of the velocity ellipsoid. For this determination we start from the expression:

$$F\,dx'dy'dz' = He^{-f/2}dx'dy'dz'$$

for the number of stars having the velocity components $x' \pm \frac{1}{2}dx'$, $y' \pm \frac{1}{2}dy'$, $z' \pm \frac{1}{2}dz'$. Here

$$f = A'x'^2 + B'y'^2 + C'z'^2$$
$$A' = 1/\sigma_1^2, \quad B' = 1/\sigma_2^2, \quad C' = 1/\sigma_3^2.$$

We now turn the system of coördinates by the formulae

$$x' = \epsilon_{11}x + \epsilon_{21}y + \epsilon_{31}z$$
$$y' = \epsilon_{12}x + \epsilon_{22}y + \epsilon_{32}z$$
$$z' = \epsilon_{13}x + \epsilon_{23}y + \epsilon_{33}z$$

so that the new z-axis has the direction (θ, ψ) which implies that

$$\epsilon_{31} = \cos\theta\cos\psi, \quad \epsilon_{32} = \cos\theta\sin\psi, \quad \epsilon_{33} = \sin\theta. \tag{28}$$

We then get

$$F\,dxdydz = He^{-f/2}dxdydz \tag{29}$$

where

$$f = Ax^2 + By^2 + Cz^2 + 2Dyz + 2Ezx + 2Fxy$$

and

$$A = A'\epsilon_{11}^2 + B'\epsilon_{12}^2 + C'\epsilon_{13}^2$$
$$B = A'\epsilon_{21}^2 + B'\epsilon_{22}^2 + C'\epsilon_{23}^2$$
$$F = A'\epsilon_{11}\epsilon_{21} + B'\epsilon_{12}\epsilon_{22} + C'\epsilon_{13}\epsilon_{23}.$$

It is not necessary to write the other coefficients.

Now integrating over all values of x and y between $-\infty$ and $+\infty$ we get

$$dz\int_{-\infty}^{+\infty}\!\!\int Fdxdy = \frac{2\pi . H}{\sigma\sqrt{A'B'C'}}e^{-\frac{z^2}{2\sigma^2}}dz = \frac{1}{\sigma\sqrt{2\pi}}.e^{-\frac{z^2}{2\sigma^2}}.dz$$

where

$$\frac{1}{\sigma^2} = \frac{A'B'C'}{B'C'\epsilon_{31}^2 + C'A'\epsilon_{32}^2 + A'B'\epsilon_{33}^2}$$

or introducing the values of A', B', C', ϵ_{31}, ϵ_{32}, ϵ_{33}

$$\sigma^2 = \sigma_1^2\cos^2\theta\cos^2\psi + \sigma_2^2\cos^2\theta\sin^2\psi + \sigma_3^2\sin^2\theta. \tag{30}$$

The polar equation of the moment ellipsoid is, however, (if s = radius vector)

$$\frac{1}{s^2} = \sigma_1^2\cos^2\theta\cos^2\psi + \sigma_2^2\cos^2\theta\sin^2\psi + \sigma_3^2\sin^2\theta. \tag{31}$$

Comparing these two equations we conclude that the dispersion in the components of the velocity of the stars in any direction *is equal to the inverse value of the radius vector of the moment ellipsoid* in the same direction.

The integration over x and y indicated above is somewhat tedious to perform and is here only alluded to. A shorter derivation of the formula (30) is given below.

Along the principal axes the dispersion is equal to the semiaxes σ_1, σ_2, and σ_3 of the velocity ellipsoid. In other directions it is, however, not equal to the radius vector of the velocity ellipsoid but always greater than it.

We have, indeed, in a direction having the direction cosines ϵ_{31}, ϵ_{32}, and ϵ_{33}

$$\frac{1}{\rho^2} = \frac{\epsilon_{31}^2}{\sigma_1^2} + \frac{\epsilon_{32}^2}{\sigma_2^2} + \frac{\epsilon_{33}^2}{\sigma_3^2}$$

$$\sigma^2 = \epsilon_{31}^2\sigma_1^2 + \epsilon_{32}^2\sigma_2^2 + \epsilon_{33}^2\sigma_3^2$$

and hence

$$\frac{\sigma^2}{\rho^2} = \epsilon_{31}^4 + \epsilon_{32}^4 + \epsilon_{33}^4 + \epsilon_{32}^2\epsilon_{33}^2\left(\frac{\sigma_2^2}{\sigma_3^2} + \frac{\sigma_3^2}{\sigma_2^2}\right)$$
$$+ \epsilon_{33}^2\epsilon_{31}^2\left(\frac{\sigma_3^2}{\sigma_1^2} + \frac{\sigma_1^2}{\sigma_3^2}\right) + \epsilon_{31}^2\epsilon_{32}^2\left(\frac{\sigma_1^2}{\sigma_2^2} + \frac{\sigma_2^2}{\sigma_1^2}\right).$$

But the quantities in parentheses in this expression are all greater than, or equal to, two. Hence we have

$$\frac{\sigma^2}{\rho^2} \geq (\epsilon_{31}^2 + \epsilon_{32}^2 + \epsilon_{33}^2)^2 = 1,$$

so that σ is always greater than ρ excepting along the three principal axes.

From the relation between the velocity ellipsoid and the moment ellipsoid we also explain why the dispersion in the radial velocities has its maximum at the same place in the sky toward which the greater axis of the velocity ellipsoid is directed (vertex and antivertex).

35. *Turning of moments.*—Through the relation between the characteristics of a frequency function and its moments, it follows that the changes in the characteristics with any alterations in the system of coördinates is fully known as soon as the corresponding changes in the moments have been determined. We have in stellar statistics to consider the movements of the stars referred to different systems of coördinates and we therefore want to know the relations between the moments by a turning of the system of coördinates.

Let us consider the two systems, I and II. The velocities in I may be x_1, y_1, z_1 and in II x_2, y_2, z_2. The moments in I we denote by μ'_{ijk} (1) and in II by μ'_{ijk} (2), so that

$$\mu'_{ijk}(1) = \iiint x_1^i y_1^j z_1^k F \, dx_1 dy_1 dz_1$$

$$\mu'_{ijk}(2) = \iiint x_2^i y_2^j z_2^k F \, dx_2 dy_2 dz_2$$

Between $(x_1y_1z_1)$ and $(x_2y_2z_2)$ may exist the relations

$$
\begin{aligned}
x_1 &= \gamma_{11}x_2+\gamma_{21}y_2+\gamma_{31}z_2,\\
y_1 &= \gamma_{12}x_2+\gamma_{22}y_2+\gamma_{32}z_2,\\
z_1 &= \gamma_{13}x_2+\gamma_{23}y_2+\gamma_{33}z_2.
\end{aligned}
\tag{32}
$$

Substituting these values in the formula for μ'_{ijk} (1) we find that the right member is directly expressed as a linear function of the moments in System II, all of the same order. Similar formulae are valid for the moments μ'_{ijk} (about any point). We thus obtain the following formulae, containing all relations to the second order.

$$
\begin{aligned}
\nu'_{100}(1) &= \gamma_{11}\nu'_{100}(2)+\gamma_{21}\nu'_{010}(2)+\gamma_{31}\nu'_{001}(2),\\
\nu'_{010}(1) &= \gamma_{12}\nu'_{100}(2)+\gamma_{22}\nu'_{010}(2)+\gamma_{32}\nu'_{001}(2),\\
\nu'_{001}(1) &= \gamma_{13}\nu'_{100}(2)+\gamma_{23}\nu'_{010}(2)+\gamma_{33}\nu'_{001}(2),
\end{aligned}
$$

$$
\begin{aligned}
\nu'_{200}(1) &= \gamma_{11}^2\nu'_{200}(2)+\gamma_{21}^2\nu'_{020}(2)+\gamma_{31}^2\nu'_{002}(2)+\\
&\quad +2\gamma_{21}\cdot\gamma_{31}\nu'_{011}(2)+2\gamma_{31}\cdot\gamma_{11}\cdot\nu'_{101}(2)+2\cdot\gamma_{11}\gamma_{21}\cdot\nu'_{110}(2)\\
\nu'_{020}(1) &= \gamma_{12}^2\cdot\nu'_{200}(2)+\gamma_{22}^2\nu'_{020}(2)+\gamma_{32}^2\nu'_{002}(2)+\\
&\quad +2\gamma_{22}\gamma_{32}\cdot\nu'_{011}(2)+2\gamma_{32}\gamma_{12}\cdot\nu'_{101}(2)+2\gamma_{12}\gamma_{22}\nu'_{110}(2)\\
\nu'_{002}(1) &= \gamma_{13}^2\cdot\nu'_{200}(2)+\gamma_{23}^2\nu'_{020}(2)+\gamma_{33}^2\nu'_{002}(2)+\\
&\quad +2\gamma_{23}\gamma_{33}\nu'_{011}(2)+2\gamma_{33}\cdot\gamma_{13}\cdot\nu'_{101}(2)+2\gamma_{13}\cdot\gamma_{23}\nu'_{110}(2)\\
\nu'_{011}(1) &= \gamma_{12}\gamma_{13}\nu'_{200}(2)+\gamma_{22}\gamma_{23}\nu'_{020}(2)+\gamma_{32}\gamma_{33}\cdot\nu'_{002}(2)+\\
&\quad +(\gamma_{22}\gamma_{33}+\gamma_{32}\gamma_{23})\nu'_{011}(2)+(\gamma_{12}\gamma_{33}+\gamma_{32}\gamma_{13})\nu'_{101}(2)\\
&\quad +(\gamma_{12}\gamma_{23}+\gamma_{22}\gamma_{13})\nu'_{110}(2)\\
\nu'_{101}(1) &= \gamma_{11}\gamma_{13}\nu'_{200}(2)+\gamma_{21}\gamma_{23}\nu'_{020}(2)+\gamma_{31}\gamma_{33}\nu'_{002}(2)+\\
&\quad +(\gamma_{21}\gamma_{33}+\gamma_{31}\gamma_{23})\nu'_{011}(2)+(\gamma_{11}\gamma_{33}+\gamma_{31}\gamma_{13})\nu'_{101}(2)+(\gamma_{11}\gamma_{23}+\gamma_{21}\gamma_{13})\nu'_{110}(2)\\
\nu'_{110}(1) &= \gamma_{11}\gamma_{12}\nu'_{200}(2)+\gamma_{21}\gamma_{22}\nu'_{020}(2)+\gamma_{31}\gamma_{32}\nu'_{002}(2)+\\
&\quad +(\gamma_{21}\gamma_{32}+\gamma_{31}\gamma_{22})\nu'_{011}(2)+(\gamma_{11}\gamma_{32}+\gamma_{31}\gamma_{12})\nu'_{101}(2)+(\gamma_{11}\gamma_{22}+\gamma_{21}\gamma_{12})\nu'_{110}(2)
\end{aligned}
$$

It may be observed that these transformation formulae *are also valid for the moments about the mean.* We may blot out all the commas on the moments and get formulae for these moments. This is evident because the relations (32) also may be written

$$
\begin{aligned}
x_1-\bar{x}_1 &= \gamma_{11}(x_2-\bar{x}_2)+\gamma_{21}(y_2-\bar{y}_2)+\gamma_{31}(z_2-\bar{z}_2)\\
y_1-\bar{y}_1 &= \gamma_{12}(x_2-\bar{x}_2)+\gamma_{22}(y_2-\bar{y}_2)+\gamma_{32}(z_2-\bar{z}_2)\\
z_1-\bar{z}_1 &= \gamma_{13}(x_2-\bar{x}_2)+\gamma_{23}(y_2-\bar{y}_2)+\gamma_{33}(z_2-\bar{z}_2)
\end{aligned}
$$

where $\bar{x}_1$, $\bar{y}_1$, $\bar{z}_1$ are the mean values of x_1, y_1, z_1 and $\bar{x}_2$, $\bar{y}_2$, $\bar{z}_2$ the mean values of x_2, y_2, z_2.

We may use these formulae, for instance, to derive our formula (30) for the dispersion in any direction. Let us start from the vertex system, having the axes parallel to the principal axes of the velocity ellipsoid. In this system we have

$$\nu_{200}(2)=\sigma_1^2;\quad \nu_{020}(2)=\sigma_2^2;\quad \nu_{002}(2)=\sigma_3^2$$

and

$$\nu_{011}(2)=0 \qquad \nu_{101}(2)=0 \qquad \nu_{110}(2)=0.$$

Using now the formulae above and supposing that the vertex system coincides with System II, we have for the dispersion in any direction (coinciding, for instance, with the Z-axis of System I) the formula

$$\sigma^2 = \nu_{002}(1) = \gamma_{13}^2\sigma_1^2 + \gamma_{23}^2\sigma_2^2 + \gamma_{33}^2\sigma_3^2 \tag{33}$$

where

$$\gamma_{13} = \cos\theta\cos\psi,\ \gamma_{23} = \cos\theta\sin\psi,\ \gamma_{33} = \sin\theta$$

which is our formula (30), really identical to the polar equation of the moment ellipsoid.

C. Velocity Ellipsoid Derived from Parallax Stars

36. We have in the following to consider three different systems of rectangular coördinates, which I shall denote as K_1, K_2, and K_3.

The system K_1 refers to the three principal axes of the velocity ellipsoid defined below. The axis of Z is directed toward the vertex.

The system K_2 is the galactic system of coördinates. It has the Z-axis directed toward the north galactic pole, the X-axis toward the descending node of the equator on the Milky Way, the Y-axis toward $l = +90$.

The system K_3 has the Z-axis directed toward the center of coördinates (regarding the coördinates employed for these centers I refer to *L. M.* II, 9, p. 69) of any of the 48 squares, in which we have divided the celestial sphere. The positive X- and Y-axes are directed toward increasing galactic longitude and latitude.

Further I shall denote all coördinates and velocities referred to K_1 by (X', Y', Z'; U', V', W', etc.); coördinates and velocities referred to K_2 by (X'', Y'', Z''; U'', V'', W'', etc.), and coördinates and velocities referred to K_3 by (X, Y, Z; U, V, W, etc.).

The relations between the velocities in the different systems may be given through the formulae:

Between K_1 and K_2

$$\begin{aligned} U' &= \epsilon_{11}U'' + \epsilon_{21}V'' + \epsilon_{31}W'' \\ V' &= \epsilon_{12}U'' + \epsilon_{22}V'' + \epsilon_{32}W'' \\ W' &= \epsilon_{13}U'' + \epsilon_{23}V'' + \epsilon_{33}W'' \end{aligned} \tag{34}$$

and

$$\begin{aligned} U'' &= \epsilon_{11}U' + \epsilon_{12}V' + \epsilon_{13}W' \\ V'' &= \epsilon_{21}U' + \epsilon_{22}V' + \epsilon_{23}W' \\ W'' &= \epsilon_{31}U' + \epsilon_{32}V' + \epsilon_{33}W' \end{aligned} \tag{35}$$

Between K_2 and K_3

$$\begin{aligned} U'' &= \gamma_{11}U + \gamma_{12}V + \gamma_{13}W \\ V'' &= \gamma_{21}U + \gamma_{22}V + \gamma_{23}W \\ W'' &= \gamma_{31}U + \gamma_{32}V + \gamma_{33}W \end{aligned} \tag{36}$$

and

$$\begin{aligned} U &= \gamma_{11}U'' + \gamma_{21}V'' + \gamma_{31}W'' \\ V &= \gamma_{12}U'' + \gamma_{22}V'' + \gamma_{32}W'' \\ W &= \gamma_{13}U'' + \gamma_{23}V'' + \gamma_{33}W'' \end{aligned} \tag{37}$$

37. For stars with known parallax the proper motions (u, v) and generally also the radial velocity (W) are known. We treat here only such stars for which this is the case so that for each one of them we know the quantities

$$\pi;\ u;\ v;\ W. \tag{38}$$

Instead of π we may use the distance (r), according to the formula

$$r = \frac{0''.206265}{\pi}. \tag{39}$$

Between the linear velocity components (U, V) and the proper motions (u, v) in the System K_3 we have the relations

$$U = ru, \quad V = rv$$

where (u, v) are to be expressed in radians per stellar year. If u and v are expressed in seconds of arc per year we have to divide by $0''.206265$ to get their values in radians per stellar year. The formulae are then

$$U = \frac{ru}{0''.206265} \qquad V = \frac{rv}{0''.206265}$$

or, according to (39).

$$U = \frac{u}{\pi}, \quad V = \frac{v}{\pi}. \tag{40}$$

With the help of these formulae we deduce from (38) the three components

$$U,\ V,\ W$$

in the system of coördinates K_3.

Now using the relations (36) we get the velocity components

$$U'',\ V'',\ W''$$

in the galactic system K_2.

If N parallax stars are given we obtain a statistical series of $3N$ values for which the moments may be computed in the usual manner.

Let N_{ijk} be the moments about the means. From the moments of the first order we get the stream velocity of the parallax stars and the corresponding coördinates of the Apex. Then we form the equation of Jacobi belonging to the moment ellipsoid

$$G(t) = \begin{vmatrix} N_{200}-t, & N_{110}, & N_{101} \\ N_{110}, & N_{020}-t, & N_{011} \\ N_{101}, & N_{011}, & N_{002}-t \end{vmatrix} = 0$$

The three roots t_1, t_2, t_3 of this equation coincide with the square of the three axes σ_1, σ_2, and σ_3 of the velocity ellipsoid, so that

$$t_1=\sigma_1^2,\ t_2=\sigma_2^2,\ t_3=\sigma_3^2.$$

The direction cosines l, m, n of these axes are obtained from

$$\frac{l}{G_{11}}=\frac{m}{G_{12}}=\frac{n}{G_{13}}=\frac{1}{\sqrt{G_{11}^2+G_{12}^2+G_{13}^2}}. \tag{41}$$

38. *Numerical application.*—In the autumn of 1922 the parallax (as well as proper motions and radial velocity) was known for 646 stars brighter than the sixth magnitude. From the proper motions the linear velocities U and V, perpendicular to the line of sight, were computed for each star according to (40) and then from U, V, and W the galactic components U'', V'', and W''. The mean values of these components were

$$\begin{aligned} U_o'' &= -3.201 \\ V_o'' &= -1.385 \\ W_o'' &= -1.413 \end{aligned}$$

which values lead to the coördinates of the Apex given in §12.

For the second moments about the mean the following values were obtained:

$$\begin{aligned} N_{200} &= +32.65 \\ N_{110} &= -\ 5.453 \\ N_{101} &= +\ 1.286 \\ N_{020} &= +18.51 \\ N_{011} &= +\ 0.270 \\ N_{002} &= +11.75 \end{aligned}$$

Hence the equation of Jacobi has the form

$$G(t)=\begin{vmatrix} +32.65-t, & -\ 5.453, & +\ 1.286 \\ -\ 5.453, & +18.51-t, & +\ 0.270 \\ +\ 1.286, & +\ 0.270, & +11.75-t \end{vmatrix}=0$$

which has the roots

$$\begin{aligned} t_1 &= +11.63=\sigma_1^2 \text{ so that } \sigma_1=+3.410 \\ t_2 &= +16.71=\sigma_2^3 \qquad\quad \sigma_2=+4.088 \\ t_3 &= +34.57=\sigma_3^2 \qquad\quad \sigma_3=+5.880 \end{aligned}$$

From (41) we get the following values of the direction cosines (l, m, n) of the three axes

$$\begin{aligned} l_1 &= -0.0724, & l_2 &= +0.3166, & l_3 &= +0.9460 \\ m_1 &= -0.0763, & m_2 &= +0.9390, & m_3 &= -0.3204 \\ n_1 &= +0.7727, & n_2 &= +0.1332, & n_3 &= +0.0495 \end{aligned}$$

which correspond to the following values in polar coördinates:

$$\begin{aligned} L_1 &= 233^\circ.06, & L_2 &= 71^\circ.36, & L_3 &= 341^\circ.29 \\ B_1 &= +83^\circ.07, & B_2 &= +7^\circ.65, & B_3 &= +2^\circ.84 \end{aligned}$$

The shortest axis (σ_1) is directed toward the galactic pole (approximately). The longest axis toward the vertex in near accordance with the result from the radial velocities and the proper motions.

D. Velocity Ellipsoid Derived from Radial Velocities

39. From the radial velocities of the stars in a square we compute the moments of the first and second order in the system K_3. These moments* are

$$N'_{001}(3) \quad \text{and} \quad N'_{002}(3)$$

According to the formulae in §35 we have the relations

$$N'_{001}(3) = \gamma_{13}N'_{100}(2) + \gamma_{23}N'_{010}(2) + \gamma_{33}N'_{001}(2) \tag{42}$$

$$\begin{aligned} N'_{002}(3) = {} & \gamma_{13}^2N'_{200}(2) + \gamma_{23}^2N'_{020}(2) + \gamma_{33}^2N'_{002}(2) \\ & + 2\gamma_{23}\gamma_{33}N'_{011}(2) + 2\gamma_{33}\gamma_{13}N'_{101}(2) + 2\gamma_{13}\gamma_{23}N'_{110}(2). \end{aligned} \tag{43}$$

For each square we get such relations. Hence we have 48 equations by which to determine the three moments of the first order in the system K_2 and 48 equations to determine the six moments of the second order in K_2. Making this determination through the method of least squares we obtain the values of

$$N'_{100}(2) \quad N'_{010}(2) \quad N'_{001}(2)$$

$$N'_{200}(2) \quad N'_{020}(2) \quad N'_{002}(2) \quad N'_{011}(2) \quad N'_{101}(2) \quad N'_{110}(2).$$

From the moments of the first order we now get the components U_o'', V_o'', W_o'' of the stream velocity:

$$U_o'' = N'_{100}(2), \quad V_o'' = N'_{010}(2), \quad W_o'' = N'_{001}(2).$$

Further, we derive, according to the formulae in §30, the moments about the means, which we denote by

$$N_{200}, \quad N_{020}, \quad N_{002}, \quad N_{011}, \quad N_{101}, \quad N_{110}$$

We may observe that we may also compute these moments directly from (43), if we change $N'_{002}(3)$ for $N_{002}(3) = \sigma^2$ in the left member. Here σ = the dispersion in W.

With these moments we now form the equation of Jacobi

$$I(t) = \begin{vmatrix} N_{200} - t, & N_{110}, & N_{101} \\ N_{110}, & N_{020} - t, & N_{011} \\ N_{101}, & N_{011}, & N_{002} - t \end{vmatrix} = 0 \tag{44}$$

The three roots of this equation are identical with the square of the three principal axes of the velocity ellipsoid:

$$\sigma_1^2, \quad \sigma_2^2, \quad \text{and} \quad \sigma_3^2.$$

* I will indicate the moments of the velocities by N_{ijk} to distinguish them from the moments of the proper motions which are indicated by ν_{ijk}.

The direction of the axes are given by the equations:

$$\frac{l}{I_{11}} = \frac{m}{I_{12}} = \frac{n}{I_{13}} = \frac{1}{\sqrt{I_{11}^2+I_{12}^2+I_{13}^2}} . \tag{45}$$

The determination of the velocity ellipsoid is then completed.

The position of the Apex is given directly through the stream components U_o'', V_o'', and W_o''. In table 9 the values of the coefficient γ_{ijk} for the different squares are given. The trigonometrical expressions of these coefficients are given in *L. M.* II, 14, §4, where we have to use the galactic coördinates (l b) instead of the equatorial coördinates (α, δ).

When the equations (42) are treated according to the method of least squares we get the equations

$$\begin{aligned}
\sum\gamma_{13}N'_{001}(3) &= N'_{100}(2)\sum\gamma_{13}^2+N'_{010}(2)\sum\gamma_{13}\gamma_{23}+N'_{001}(2)\sum\gamma_{13}\gamma_{33}\\
\sum\gamma_{23}N'_{001}(3) &= N'_{100}(2)\sum\gamma_{23}\gamma_{13}+N'_{010}(2)\sum\gamma_{23}^2+N'_{001}(2)\sum\gamma_{23}\gamma_{33}\\
\sum\gamma_{33}N'_{001}(3) &= N'_{100}(2)\sum\gamma_{33}\gamma_{13}+N'_{010}(2)\sum\gamma_{33}\gamma_{23}+N'_{001}(2)\sum\gamma_{33}^2
\end{aligned}$$

supposing that the same weight be given to all the squares. From the values of the direction cosines γ_{ij} given in table 9 we find, however, that

$$\sum\gamma_{23}\gamma_{33} = \sum\gamma_{33}\gamma_{13} = \sum\gamma_{13}\gamma_{23} = 0 \tag{46}$$

Further we have

$$\begin{aligned}
\sum\gamma_{13}^2 &= +16.2324\\
\sum\gamma_{23}^2 &= +16.3552\\
\sum\gamma_{33}^2 &= +15.4132
\end{aligned} \tag{46*}$$

the normal equations take the form:

$$\begin{aligned}
16.2324\ N'_{100}(2) &= \sum\gamma_{13}N'_{001}(3)\\
16.3552\ N'_{010}(2) &= \sum\gamma_{23}N'_{001}(3)\\
15.4132\ N'_{001}(2) &= \sum\gamma_{33}N'_{001}(3)
\end{aligned} \tag{47}$$

Treating in like manner the equations (43) for the moments of the second order we observe that

$$\begin{aligned}
\sum\gamma_{13}^2\gamma_{23}\gamma_{33} &= \sum\gamma_{13}^3\gamma_{23} = \sum\gamma_{13}^3\gamma_{33} = 0\\
\sum\gamma_{13}\gamma_{23}^2\gamma_{33} &= \sum\gamma_{13}\gamma_{23}^3 = \sum\gamma_{23}^3\gamma_{33} = 0\\
\sum\gamma_{13}\gamma_{23}\gamma_{33}^2 &= \sum\gamma_{13}\gamma_{33}^3 = \sum\gamma_{23}\gamma_{33}^3 = 0
\end{aligned} \tag{48}$$

TABLE 9

Direction cosines for the different squares

	γ_{11}	γ_{12}	γ_{13}	γ_{21}	γ_{22}	γ_{23}	γ_{32}	γ_{33}
A_1	−1.0000	0.0000	0.0000	0.0000	−0.9848	+0.1736	+0.1736	+0.9848
A_2	+1.0000	0.0000	0.0000	0.0000	+0.9848	−0.1736	+0.1736	+0.9848
B_1	−0.3090	−0.6737	+0.6714	+0.9511	−0.2189	+0.2181	+0.7059	+0.7083
B_2	−0.8090	−0.4163	+0.4149	+0.5878	−0.5730	+0.5711	+0.7059	+0.7083
B_3	−1.0000	0.0000	0.0000	0.0000	−0.7083	+0.7059	+0.7059	+0.7083
B_4	−0.8090	+0.4163	−0.4149	−0.5878	−0.5730	+0.5711	+0.7059	+0.7083
B_5	−0.3090	+0.6737	−0.6714	−0.9511	−0.2189	+0.2181	+0.7059	+0.7083
B_6	+0.3090	+0.6737	−0.6714	−0.9511	+0.2189	−0.2181	+0.7059	+0.7083
B_7	+0.8090	+0.4163	−0.4149	−0.5878	+0.5730	−0.5711	+0.7059	+0.7083
B_8	+1.0000	0.0000	0.0000	0.0000	+0.7083	−0.7059	+0.7059	+0.7083
B_9	+0.8090	−0.4163	+0.4149	+0.5878	+0.5730	−0.5711	+0.7059	+0.7083
B_{10}	+0.3090	−0.6737	+0.6714	+0.9511	+0.2189	−0.2181	+0.7059	+0.7083
C_1	−0.2588	−0.2415	+0.9353	+0.9659	−0.0647	+0.2506	+0.9683	+0.2500
C_2	−0.7071	−0.1768	+0.6847	+0.7071	−0.1768	+0.6847	+0.9683	+0.2500
C_3	−0.9659	−0.0647	+0.2506	+0.2588	−0.2415	+0.9353	+0.9683	+0.2500
C_4	−0.9659	+0.0647	−0.2506	−0.2588	−0.2415	+0.9353	+0.9683	+0.2500
C_5	−0.7071	+0.1768	−0.6847	−0.7071	−0.1768	+0.6847	+0.9683	+0.2500
C_6	−0.2588	+0.2415	−0.9353	−0.9659	−0.0647	+0.2506	+0.9683	+0.2500
C_7	+0.2588	+0.2415	−0.9353	−0.9659	+0.0647	−0.2506	+0.9683	+0.2500
C_8	+0.7071	+0.1768	−0.6847	−0.7071	+0.1768	−0.6847	+0.9683	+0.2500
C_9	+0.9659	+0.0647	−0.2506	−0.2588	+0.2415	−0.9353	+0.9683	+0.2500
C_{10}	+0.9659	−0.0647	+0.2506	+0.2588	+0.2415	−0.9353	+0.9683	+0.2500
C_{11}	+0.7071	−0.1768	+0.6847	+0.7071	+0.1768	−0.6847	+0.9683	+0.2500
C_{12}	+0.2588	−0.2415	+0.9353	+0.9659	+0.0647	−0.2506	+0.9683	+0.2500
D_1	−0.2588	+0.2415	+0.9353	+0.9659	+0.0647	+0.2506	+0.9683	−0.2500
D_2	−0.7071	+0.1768	+0.6847	+0.7071	+0.1768	+0.6847	+0.9683	−0.2500
D_3	−0.9659	+0.0647	+0.2506	+0.2588	+0.2415	+0.9353	+0.9683	−0.2500
D_4	−0.9659	−0.0647	−0.2506	−0.2588	+0.2415	+0.9353	+0.9683	−0.2500
D_5	−0.7071	−0.1768	−0.6847	−0.7071	+0.1768	+0.6847	+0.9683	−0.2500
D_6	−0.2588	−0.2415	−0.9353	−0.9659	+0.0647	+0.2506	+0.9683	−0.2500
D_7	+0.2588	−0.2415	−0.9353	−0.9659	−0.0647	−0.2506	+0.9683	−0.2500
D_8	+0.7071	−0.1768	−0.6847	−0.7071	−0.1768	−0.6847	+0.9683	−0.2500
D_9	+0.9659	−0.0647	−0.2506	−0.2588	−0.2415	−0.9353	+0.9683	−0.2500
D_{10}	+0.9659	+0.0647	+0.2506	+0.2588	−0.2415	−0.9353	+0.9683	−0.2500
D_{11}	+0.7071	+0.1768	+0.6847	+0.7071	−0.1768	−0.6847	+0.9683	−0.2500
D_{12}	+0.2588	+0.2415	+0.9353	+0.9659	−0.0647	−0.2506	+0.9683	−0.2500
E_1	−0.3090	+0.6737	+0.6714	+0.9511	+0.2189	+0.2181	+0.7059	−0.7083
E_2	−0.8090	+0.4163	+0.4149	+0.5878	+0.5730	+0.5711	+0.7059	−0.7083
E_3	−1.0000	0.0000	0.0000	0.0000	+0.7083	+0.7059	+0.7059	−0.7083
E_4	−0.8090	−0.4163	−0.4149	−0.5878	+0.5730	+0.5711	+0.7059	−0.7083
E_5	−0.3090	−0.6737	−0.6714	−0.9511	+0.2189	+0.2181	+0.7059	−0.7083
E_6	+0.3090	−0.6737	−0.6714	−0.9511	−0.2189	−0.2181	+0.7059	−0.7083
E_7	+0.8090	−0.4163	−0.4149	−0.5878	−0.5730	−0.5711	+0.7059	−0.7083
E_8	+1.0000	0.0000	0.0000	0.0000	−0.7083	−0.7059	+0.7059	−0.7083
E_9	+0.8090	+0.4163	+0.4149	+0.5878	−0.5730	−0.5711	+0.7059	−0.7083
E_{10}	+0.3090	+0.6737	+0.6714	+0.9511	−0.2189	−0.2181	+0.7059	−0.7083
F_1	−1.0000	0.0000	0.0000	0.0000	+0.9848	+0.1736	+0.1736	−0.9848
F_2	+1.0000	0.0000	0.0000	0.0000	−0.9848	−0.1736	+0.1736	−0.9848

and further

$$
\begin{aligned}
&\sum\gamma_{13}^4 = +9.7744 \qquad \sum\gamma_{23}^2\gamma_{33}^2 = +3.3208\\
&\sum\gamma_{23}^4 = +9.7784 \qquad \sum\gamma_{33}^2\gamma_{13}^2 = +3.2032\\
&\sum\gamma_{33}^4 = +8.8896 \qquad \sum\gamma_{13}^2\gamma_{23}^2 = +3.2576
\end{aligned} \tag{48*}
$$

so that the normal equations for the moments take the form:

$$
\begin{aligned}
9.7744\, N'_{200}(2) + 3.2576\, N'_{020}(2) + 3.2032\, N'_{002}(2) &= \sum\gamma_{13}^2 N'_{002}(3)\\
3.2576\, N'_{200}(2) + 9.7784\, N'_{020}(2) + 3.3208\, N'_{002}(2) &= \sum\gamma_{23}^2 N'_{002}(3)\\
3.2032\, N'_{200}(2) + 3.3208\, N'_{020}(2) + 8.8896\, N'_{002}(2) &= \sum\gamma_{33}^2 N'_{002}(3)
\end{aligned} \tag{49}
$$

$$
\begin{aligned}
6.6416\, N'_{011}(2) &= \sum\gamma_{23}\gamma_{33} N'_{002}(3)\\
6.4064\, N'_{101}(2) &= \sum\gamma_{33}\gamma_{13} N'_{002}(3)\\
6.5152\, N'_{110}(2) &= \sum\gamma_{13}\gamma_{23} N'_{002}(3)
\end{aligned} \tag{50}
$$

The coefficients of the unknown quantities in (49) being constants, their solutions may be written in a more simple form. Putting

$$
\begin{aligned}
a &= \sum\gamma_{13}^2 N'_{002}(3)\\
b &= \sum\gamma_{23}^2 N'_{002}(3)\\
c &= \sum\gamma_{33}^2 N'_{002}(3)
\end{aligned} \tag{51}
$$

we get

$$
\begin{aligned}
N'_{200}(2) &= +0.123113\, a - 0.029719\, b - 0.033260\, c\\
N'_{020}(2) &= -0.029719\, a + 0.124299\, b - 0.035725\, c\\
N'_{002}(2) &= -0.033260\, a - 0.035725\, b + 0.137821\, c
\end{aligned} \tag{52}
$$

The moments of the second order in the galactic system of coördinates K_2 are thus directly expressed in the moments $N'_{002}(3)$.

40. *Numerical application.*—The 1986 stars for which radial velocities were known in the autumn of 1922 were treated according to the method of the preceding article. For the moments $N'_{001}(3)$ and $N'_{002}(3)$ the values obtained are noted in table 10 for the different squares.

Using the formulae (46) we now get

$$
\begin{aligned}
N'_{100}(2) &= -3.3706\\
N'_{010}(2) &= -1.4466\\
N'_{001}(2) &= -1.6191
\end{aligned}
$$

TABLE 10

Moments of the radial velocities of the first and second order in different squares

	N	$N'_{001}(3)$	$N'_{002}(3)$		N	$N'_{001}(3)$	$N'_{002}(3)$
GA_1	26	−1.712	+14.37	GD_1	53	−3.473	+21.74
GA_2	35	−0.738	+10.08	GD_2	53	−2.235	+15.53
				GD_3	53	−1.593	+10.19
GB_1	50	−3.326	+27.28	GD_4	67	−0.471	+12.26
GB_2	53	−2.785	+20.11	GD_5	60	+3.142	+20.39
GB_3	48	−1.366	+ 9.77	GD_6	82	+4.544	+29.45
GB_4	46	−0.442	+16.97	GD_7	46	+4.945	+37.10
GB_5	33	+1.099	+13.91	GD_8	49	+4.075	+21.20
GB_6	38	+1.838	+20.85	GD_9	70	+2.799	+17.42
GB_7	21	+0.882	+13.69	GD_{10}	33	+0.730	+ 9.27
GB_8	17	−1.303	+ 8.94	GD_{11}	47	−0.388	+18.13
GB_9	19	−1.150	+14.82	GD_{12}	40	−1.814	+17.44
GB_{10}	42	−3.968	+33.27				
GC_1	55	−3.989	+26.16	GE_1	24	−0.875	+12.92
GC_2	74	−3.350	+18.90	GE_2	34	−1.299	+ 9.96
GC_3	56	−2.675	+21.10	GE_3	37	−0.229	+ 8.27
GC_4	50	−1.218	+12.40	GE_4	40	+1.532	+14.91
GC_5	41	+0.839	+13.71	GE_5	35	+2.465	+22.07
GC_6	52	+2.209	+23.36	GE_6	27	+2.481	+16.00
GC_7	27	+3.721	+22.47	GE_7	33	+3.232	+19.95
GC_8	25	+4.085	+28.40	GE_8	29	+2.053	+12.89
GC_9	37	+1.677	+10.40	GE_9	27	+1.974	+17.55
GC_{10}	45	+0.172	+ 9.65	GE_{10}	33	+0.724	+13.36
GC_{11}	47	−2.241	+14.43				
GC_{12}	33	−2.503	+23.46	GF_1	19	+1.109	+15.22
				GF_2	25	+2.107	+12.82

These moments coincide with the components U_0'', V_0'', W_0'' of the stream-velocity and lead to the velocity of the sun and the coördinates of the Apex that are given in §10.

From the moments of the second order we obtain, according to (52) and (50)

$$N'_{200}(2) = +27.3801 \qquad N'_{011}(2) = +1.8326$$
$$N'_{020}(2) = +11.2986 \qquad N'_{101}(2) = +4.6145$$
$$N'_{002}(2) = +13.3192 \qquad N'_{110}(2) = +2.1579$$

With the formulae in §30 we now obtain for the moments N_{ijk} about the means the values:

$$N_{200} = +16.0192 \qquad N_{011} = -0.5096$$
$$N_{020} = +\ 9.2059 \qquad N_{101} = -0.8428$$
$$N_{002} = +10.6977 \qquad N_{110} = -2.7180$$

The equation of Jacobi takes the form

$$F(t)=\begin{vmatrix} +16.0192-t, & -2.7180, & -0.8428 \\ -2.7180, & +9.2059-t, & -0.5096 \\ -0.8428, & -0.5096, & +10.6977-t \end{vmatrix}=0$$

Developing into powers of t we get

$$-t^3+35.9228\,t^2-408.9643\,t+1485.5390=0$$

The solution of this equation is made graphically and leads to the roots:

$$\begin{array}{ll} t_1=+10.847 \text{ so that } & \sigma_1=+3.294 \\ t_2=+8.036 & \sigma_2=+2.835 \\ t_3=+17.033 & \sigma_3=+4.127 \end{array}$$

The direction cosines of the three axes are calculated according to the equation (45). The values obtained were given in §16.

E. Apex and Velocity Ellipsoid Derived from Proper Motions

41. *Proper motions of the stars used for discussing their linear velocities.*— Let x and y be two variates and F (x, y) their frequency function, so that

$$F\,dx\,dy$$

is the probability that x lies between the limits $x \pm \frac{1}{2}\,dx$ and y simultaneously between the limits $y \pm \frac{1}{2}\,dy$, supposing that

$$\int_{-\infty}^{+\infty}\int F\,dx\,dy=1$$

The question is: how great is the mean value of the product xy? We have

$$\begin{aligned} \overline{xy} &= \int_{-\infty}^{+\infty}\int xy\,F\,dx\,dy \\ &= \int_{-\infty}^{+\infty}\int (x-\bar{x}+\bar{x})\,(y-\bar{y}+\bar{y})\,F\,dx\,dy \\ &= \bar{x}.\bar{y}+\int\int (x-\bar{x})\,(y-\bar{y})\,F\,dx\,dy. \end{aligned}$$

The latter integral is equal to the moment ν_{11} of x and y taken about their means, so that

$$\overline{xy}=\bar{x}.\bar{y}+\nu_{11}$$

Instead of ν_{11} we may also write $r_{xy}\sigma_x\sigma_y$, r_{xy} being the coefficient of correlation and σ_x and σ_y the dispersions of x and y.

Hence we have

(53) $$\overline{xy} = \bar{x}.\bar{y} + r_{xy}\sigma_x.\sigma_y$$

We conclude that the mean value of a product of two variates is equal to the product of the means of the two variates plus a term proportional to the coefficient of correlation between the variates.

If the variates are independent of each other we have $r=0$ and the latter term vanishes.

This elementary theorem may be taken into consideration when the connection between the proper motions of the stars and their linear velocities is discussed. We have, indeed, between the proper motions u and v in galactic longitude and latitude and the components U and V in the system K_3 of the velocity of a star, the relations

(54) $$ru = U,\ rv = V$$

so that, according to (53),

$$\overline{U} = \overline{ru} = \bar{r}.\bar{u} + r_{r,u}\sigma_r\sigma_u$$

The correlation between r and u is, however, large and unknown. Since it is impossible to neglect the second term in this expression we write (54) in the following form

$$u = \frac{1}{r}U;\quad v = \frac{1}{r}V$$

and obtain then

(55) $$\bar{u} = M\left(\frac{1}{r}\right).\bar{U} + r_{\frac{1}{r}u}.\sigma_{\frac{1}{r}}.\sigma_u$$

The correlation between $1/r$ and U is, indeed, also unknown, though it may be determined with some approximation. We have, however, reason to believe that this correlation is rather small, so that we may as a good approximation put

$$\bar{u} = M\left(\frac{1}{r}\right).\bar{U}$$

or using the notation

$$\vartheta_1 = M\left(\frac{1}{r}\right)$$

(56) $$\bar{u} = \vartheta_1.\bar{U};\qquad \bar{v} = \vartheta_1\bar{V}$$

Considering more generally the moment of the order $i+j$ and putting

$$\nu'_{ij0} = \int_{-\infty}^{\infty}\int u^i v^j F\, du dv$$

$$N'_{ij0}(3) = \int_{-\infty}^{\infty}\int U^i.V^j.F\, d\,U\, d\,V$$

we evidently obtain

$$\nu'_{ijo}=\vartheta_{i+j}N'_{ijo}(3) \tag{57}$$

where

$$\vartheta_{i+j}=M\left(\frac{1}{r^{i+j}}\right). \tag{58}$$

Through (57) ν'_{ijo} may be computed from N'_{ijo} or inversely as soon as ϑ_{i+j} is known. We may also in this manner compute the value of ϑ_{i+j} if the values of ν'_{ijo} and N'_{ijo} are given.

42. Let us now consider how the Apex may be determined from the proper motions of the stars in the system K_3.

We start from the equations

$$\begin{aligned} U' &= \gamma_{11}U''+\gamma_{21}V''+\gamma_{31}W'' \\ V' &= \gamma_{12}U''+\gamma_{22}V''+\gamma_{32}W''. \end{aligned}$$

Taking the means and observing that

$$\begin{aligned} \bar{U} &= N'_{100}(3)=\frac{1}{\vartheta_1}\nu'_{100}=\frac{1}{\vartheta_1}\cdot\bar{u} \\ \bar{U}'' &= N'_{100}(2) \end{aligned}$$

and similar relations for the other components, we obtain the relations

$$\begin{aligned} \bar{u} &= \gamma_{11}\vartheta_1N'_{100}(2)+\gamma_{21}\vartheta_1N'_{010}(2)+\gamma_{31}\vartheta_1N'_{001}(2) \\ \bar{v} &= \gamma_{12}\vartheta_1N'_{100}(2)+\gamma_{22}\vartheta_1N'_{010}(2)+\gamma_{32}\vartheta_1N'_{001}(2) \end{aligned} \tag{59}$$

Solving these equations according to the method of least squares and observing that

$$\begin{aligned} \sum\gamma_{21}\gamma_{31}=\sum\gamma_{31}\gamma_{11}=\sum\gamma_{11}\gamma_{21}=0 \\ \sum\gamma_{22}\gamma_{32}=\sum\gamma_{32}\gamma_{12}=\sum\gamma_{12}\gamma_{22}=0 \end{aligned} \tag{60}$$

we obtain from the proper motions in galactic longitude

$$\begin{aligned} \vartheta_1N'_{100}(2)\sum\gamma_{11}^2 &= \sum\gamma_{11}\bar{u} \\ \vartheta_1N'_{010}(2)\sum\gamma_{21}^2 &= \sum\gamma_{21}\bar{u} \\ \vartheta_1N'_{001}(2)\sum\gamma_{31}^2 &= \sum\gamma_{31}\bar{u} \end{aligned}$$

and from the proper motions in galactic latitude

$$\begin{aligned} \vartheta_1N'_{100}(2)\sum\gamma_{12}^2 &= \sum\gamma_{12}\bar{v} \\ \vartheta_1N'_{010}(2)\sum\gamma_{22}^2 &= \sum\gamma_{22}\bar{v} \\ \vartheta_1N'_{001}(2)\sum\gamma_{32}^2 &= \sum\gamma_{32}\bar{v} \end{aligned}$$

Observing now that

$$
\begin{aligned}
&\sum\gamma_{11}{}^2 = +26.0000; \quad \sum\gamma_{12}{}^2 = +\ 5.7674 \\
&\sum\gamma_{21}{}^2 = +22.0000; \quad \sum\gamma_{22}{}^2 = +\ 9.6468 \\
&\sum\gamma_{31}{}^2 = \quad 0.0000; \quad \sum\gamma_{32}{}^2 = +32.5859
\end{aligned} \tag{60*}
$$

we obtain the equations: from longitudes

$$
\begin{aligned}
26.0000\ \vartheta_1 N'_{100}(2) &= \sum\gamma_{11}\bar{u} \\
22.0000\ \vartheta_1 N'_{010}(2) &= \sum\gamma_{21}\bar{u}
\end{aligned} \tag{61}
$$

and from the latitudes

$$
\begin{aligned}
5.7674\ \vartheta_1 N'_{100}(2) &= \sum\gamma_{12}\bar{v} \\
9.6468\ \vartheta_1 N'_{010}(2) &= \sum\gamma_{22}\bar{v} \\
32.5859\ \vartheta_1 N'_{001}(2) &= \sum\gamma_{32}\bar{v}
\end{aligned} \tag{62}
$$

and combining both

$$
\begin{aligned}
31.7674\ \vartheta_1 N'_{100}(2) &= \sum\gamma_{11}\bar{u} + \sum\gamma_{12}\bar{v} \\
31.6468\ \vartheta_1 N'_{010}(2) &= \sum\gamma_{21}\bar{u} + \sum\gamma_{22}\bar{v} \\
32.5859\ \vartheta_1 N'_{001}(2) &= \sum\gamma_{32}\bar{v}
\end{aligned} \tag{63}
$$

Putting now

$$
\begin{aligned}
N'_{100}(2) &= -\Omega \cos B \cos L \\
N'_{010}(2) &= -\Omega \cos B \sin L \\
N'_{001}(2) &= -\Omega \sin B
\end{aligned} \tag{64}
$$

we obtain the values of $\vartheta_1\ \Omega$, B, and L. The coördinates B and L of the Apex are thus obtained from the proper motions of the stars. Here the velocity Ω of the sun is not obtained but the value of $\vartheta_1\ \Omega$ is determined. Comparing this value with that found for Ω from the radial velocities we get the value of ϑ_1.

The value of ϑ_1 may also be determined for each square by comparing the mean value of $\bar{\mu}$ for each square with the stream velocity ω_o obtained from the radial velocities. Putting

$$\bar{\mu} = \sqrt{\bar{u}^2 + \bar{v}^2}; \quad \omega_o = \sqrt{U_o{}^2 + V_o{}^2}$$

we have

$$\bar{\mu} = \vartheta_1 \omega_o \tag{65}$$

The value of ϑ_1 multiplied by 0.206265 gives the mean parallax $\bar{\pi}$ of the stars for each square. These values have been given in table 7.

43. *The velocity ellipsoid determined from the proper motions.*—We have here to turn the moments of the second order from the system K_2 to the system K_3. Using the formulae of §35 we have

$$
\begin{aligned}
N'_{200}(3) &= \gamma_{11}^2 N'_{200}(2) + \gamma_{21}^2 N'_{020}(2) + \gamma_{31}^2 N'_{002}(2) \\
&\quad + 2\gamma_{21}\gamma_{31} N'_{011}(2) + 2\gamma_{31}\gamma_{11} N'_{101}(2) + 2\gamma_{11}\gamma_{21} N'_{110}(2) \\
N'_{020}(3) &= \gamma_{12}^2 N'_{200}(2) + \gamma_{22}^2 N'_{020}(2) + \gamma_{32}^2 N'_{002}(2) \\
&\quad + 2\gamma_{22}\gamma_{32} N'_{011}(2) + 2\gamma_{32}\gamma_{12} N'_{101}(2) + 2\gamma_{12}\gamma_{22} N'_{110}(2) \\
N'_{110}(3) &= \gamma_{11}\gamma_{12} N'_{200}(2) + \gamma_{21}\gamma_{22} N'_{020}(2) + \gamma_{31}\gamma_{32} N'_{002}(2) \\
&\quad + (\gamma_{31}\gamma_{22} + \gamma_{21}\gamma_{32}) N'_{011}(2) + (\gamma_{31}\gamma_{12} + \gamma_{11}\gamma_{32}) N'_{101}(2) + (\gamma_{11}\gamma_{22} + \gamma_{21}\gamma_{12}) N'_{110}(2)
\end{aligned}
\tag{66}
$$

The moments N_{200}, N_{020}, and N_{110} of the linear velocities in the system K_3 are not known but only the moments ν'_{200}, ν'_{020}, and ν'_{110} of the proper motions. We have now according to §41 the relations

$$
\begin{aligned}
\nu'_{200}(3) &= \vartheta_2 N'_{200}(3) \\
\nu'_{020}(3) &= \vartheta_2 N'_{020}(3) \\
\nu'_{110}(3) &= \vartheta_2 N'_{110}(3)
\end{aligned}
$$

so that after multiplying by ϑ_2 the moments $\nu'_{ijo}(3)$ are explained as linear functions of ϑ_2 multiplied by the moments in K_2. Treating (66) according to the methods of least squares and observing that the following relations exist*

$$
\begin{aligned}
&\sum \gamma_{11}^2\gamma_{21}\gamma_{31} = \sum \gamma_{11}^3\gamma_{21} = \sum \gamma_{11}^3\gamma_{31} = 0 \\
&\sum \gamma_{11}\gamma_{21}^2\gamma_{31} = \sum \gamma_{11}\gamma_{21}^3 = \sum \gamma_{21}^3\gamma_{31} = 0 \\
&\sum \gamma_{11}\gamma_{21}\gamma_{31}^2 = \sum \gamma_{11}\gamma_{31}^3 = \sum \gamma_{21}\gamma_{31}^3 = 0
\end{aligned}
\tag{67}
$$

$$
\begin{aligned}
&\sum \gamma_{12}^2\gamma_{22}\gamma_{32} = \sum \gamma_{12}^3\gamma_{22} = \sum \gamma_{12}^3\gamma_{32} = 0 \\
&\sum \gamma_{12}\gamma_{22}^2\gamma_{32} = \sum \gamma_{12}\gamma_{22}^3 = \sum \gamma_{22}^3\gamma_{32} = 0 \\
&\sum \gamma_{12}\gamma_{22}\gamma_{32}^2 = \sum \gamma_{12}\gamma_{32}^3 = \sum \gamma_{22}\gamma_{32}^3 = 0
\end{aligned}
\tag{68}
$$

$$
\begin{aligned}
&\gamma_{11}\gamma_{31}\gamma_{12}\gamma_{32} = \gamma_{11}\gamma_{31}\gamma_{12}\gamma_{22} = \gamma_{11}\gamma_{21}\gamma_{12}\gamma_{32} = \gamma_{11}\gamma_{31}\gamma_{12}^2 = \gamma_{11}^2\gamma_{12}\gamma_{32} = \gamma_{11}^2\gamma_{12}\gamma_{22} = \gamma_{11}\gamma_{21}\gamma_{12}^2 = 0 \\
&\gamma_{21}\gamma_{31}\gamma_{22}\gamma_{32} = \gamma_{21}\gamma_{31}\gamma_{22}^2 = \gamma_{21}^2\gamma_{22}\gamma_{32} = \gamma_{21}\gamma_{31}\gamma_{12}\gamma_{22} = \gamma_{11}\gamma_{21}\gamma_{22}\gamma_{32} = \gamma_{11}\gamma_{21}\gamma_{22}^2 = \gamma_{21}^2\gamma_{12}\gamma_{22} = 0 \\
&\gamma_{31}^2\gamma_{22}\gamma_{32} = \gamma_{21}\gamma_{31}\gamma_{32}^2 = \gamma_{31}^2\gamma_{12}\gamma_{32} = \gamma_{11}\gamma_{31}\gamma_{32}^2 = \gamma_{11}\gamma_{31}\gamma_{22}\gamma_{32} = \gamma_{21}\gamma_{31}\gamma_{12}\gamma_{32} = 0 \\
&\gamma_{31}^2\gamma_{12}\gamma_{22} = \gamma_{11}\gamma_{31}\gamma_{22}^2 = \gamma_{11}\gamma_{21}\gamma_{32}^2 = \gamma_{21}^2\gamma_{12}\gamma_{32} = \gamma_{21}\gamma_{31}\gamma_{12}^2 = \gamma_{11}^2\gamma_{22}\gamma_{32} = 0
\end{aligned}
\tag{69}
$$

all these products being added over all squares we obtain the following normal equations for determining the moments in K_2:

---

*It is found that all sums of the powers of the γ-coefficients (of the fourth order) that are not pure quadratic terms vanish, excepting the sum $\sum \gamma_{11}\gamma_{12}\gamma_{21}\gamma_{22}$, which has the value

$$\sum \gamma_{11}\gamma_{12}\gamma_{21}\gamma_{22} = -1.4418.$$

$$\sum \gamma_{11}^2 \nu'_{200} = \vartheta_2 N'_{200}(2) \sum \gamma_{11}^4 + \vartheta_2 N'_{020}(2) \sum \gamma_{11}^2 \gamma_{21}^2 + \vartheta_2 N'_{002}(2) \sum \gamma_{11}^2 \gamma_{31}^2$$

(70) $$\sum \gamma_{21}^2 \nu'_{200} = \vartheta_2 N'_{200}(2) \sum \gamma_{21}^2 \gamma_{11}^2 + \vartheta_2 N'_{020}(2) \sum \gamma_{21}^4 + \vartheta_2 N'_{002}(2) \sum \gamma_{21}^2 \gamma_{31}^2$$

$$\sum \gamma_{31}^2 \nu'_{200} = \vartheta_2 N'_{200}(2) \sum \gamma_{31}^2 \gamma_{11}^2 + \vartheta_2 N'_{020}(2) \sum \gamma_{31}^2 \gamma_{21}^2 + \vartheta_2 N'_{002}(2) \sum \gamma_{31}^4$$

$$\sum \gamma_{21}\gamma_{31} \nu'_{200} = 2\vartheta_2 N'_{011}(2) \sum \gamma_{21}^2 \gamma_{31}^2$$

(71) $$\sum \gamma_{31}\gamma_{11} \nu'_{200} = 2\vartheta_2 N'_{101}(2) \sum \gamma_{31}^2 \gamma_{11}^2$$

$$\sum \gamma_{11}\gamma_{21} \nu'_{200} = 2\vartheta_2 N'_{110}(2) \sum \gamma_{11}^2 \gamma_{21}^2$$

$$\sum \gamma_{12}^2 \nu'_{020} = \vartheta_2 N'_{200}(2) \sum \gamma_{12}^4 + \vartheta_2 N'_{020}(2) \sum \gamma_{12}^2 \gamma_{22}^2 + \vartheta_2 N'_{002}(2) \sum \gamma_{12}^2 \gamma_{32}^2$$

(72) $$\sum \gamma_{22}^2 \nu'_{020} = \vartheta_2 N'_{200}(2) \sum \gamma_{22}^2 \gamma_{12}^2 + \vartheta_2 N'_{020}(2) \sum \gamma_{22}^4 + \vartheta_2 N'_{002}(2) \sum \gamma_{22}^2 \gamma_{32}^2$$

$$\sum \gamma_{32}^2 \nu'_{020} = \vartheta_2 N'_{200}(2) \sum \gamma_{32}^2 \gamma_{12}^2 + \vartheta_2 N'_{020}(2) \sum \gamma_{32}^2 \gamma_{22}^2 + \vartheta_2 N'_{002}(2) \sum \gamma_{32}^4$$

$$\sum \gamma_{22}\gamma_{32} \nu'_{020} = 2\vartheta_2 N'_{011}(2) \sum \gamma_{22}^2 \gamma_{32}^2$$

(73) $$\sum \gamma_{32}\gamma_{12} \nu'_{020} = 2\vartheta_2 N'_{101}(2) \sum \gamma_{32}^2 \gamma_{12}^2$$

$$\sum \gamma_{12}\gamma_{22} \nu'_{020} = 2\vartheta_2 N'_{110}(2) \sum \gamma_{12}^2 \gamma_{22}^2$$

$$\sum \gamma_{11}\gamma_{12} \nu'_{110} = \vartheta_2 N'_{200}(2) \sum \gamma_{11}^2 \gamma_{12}^2 + \vartheta_2 N'_{020}(2) \sum \gamma_{11}\gamma_{12}\gamma_{21}\gamma_{22}$$

(74) $$\sum \gamma_{21}\gamma_{22} \nu'_{110} = \vartheta_2 N'_{200}(2) \sum \gamma_{11}\gamma_{12}\gamma_{21}\gamma_{22} + \vartheta_2 N'_{020}(2) \sum \gamma_{21}^2 \gamma_{22}^2$$

$$\sum \gamma_{31}\gamma_{32} \nu'_{110} = \vartheta_2 N'_{002}(2) \sum \gamma_{31}^2 \gamma_{32}^2$$

$$\sum (\gamma_{21}\gamma_{32} + \gamma_{31}\gamma_{22}) \nu'_{110} = \vartheta_2 N'_{011}(2) \cdot \sum (\gamma_{21}\gamma_{32} + \gamma_{31}\gamma_{22})^2$$

(75) $$\sum (\gamma_{11}\gamma_{32} + \gamma_{31}\gamma_{12}) \nu'_{110} = \vartheta_2 N'_{101}(2) \cdot \sum (\gamma_{11}\gamma_{32} + \gamma_{31}\gamma_{12})^2$$

$$\sum (\gamma_{11}\gamma_{22} + \gamma_{21}\gamma_{12}) \nu'_{110} = \vartheta_2 N'_{110}(2) \cdot \sum (\gamma_{11}\gamma_{22} + \gamma_{21}\gamma_{12})^2$$

Summing the equations (70) and (72), and observing that $\gamma_{31} = 0$, we obtain:

$$\sum \nu'_{200} = \vartheta_2 N'_{200}(2) \sum \gamma_{11}^2 + \vartheta_2 N'_{020}(2) \sum \gamma_{21}^2$$

$$\sum \nu'_{020} = \vartheta_2 N'_{200}(2) \sum \gamma_{12}^2 + \vartheta_2 N'_{020}(2) \sum \gamma_{22}^2 + \vartheta_2 N'_{002}(2) \sum \gamma_{32}^2$$

The sum of the three equations (74) gives the identity $0 = 0$.

44. In numerical computations we have to use the following values of the sums of the powers of the direction cosines γii:

$$
\begin{aligned}
&\textstyle\sum \gamma_{11}^4 = +20.5000, && \textstyle\sum \gamma_{11}^2\gamma_{21}^2 = +\ 5.5000\\
&\textstyle\sum \gamma_{21}^4 = +16.5000, && \textstyle\sum \gamma_{12}^2\gamma_{22}^2 = +\ 0.6411\\
&\textstyle\sum \gamma_{12}^4 = +\ 1.9233, && \textstyle\sum \gamma_{12}^2\gamma_{32}^2 = +\ 3.2030\\
&\textstyle\sum \gamma_{22}^4 = +\ 5.6857, && \textstyle\sum \gamma_{22}^2\gamma_{32}^2 = +\ 3.3200\\
&\textstyle\sum \gamma_{32}^4 = +26.0629, && \textstyle\sum \gamma_{21}^2\gamma_{32}^2 = +16.2326\\
&\qquad\qquad\quad\ \ \text{———} && \textstyle\sum \gamma_{11}^2\gamma_{32}^2 = +16.3532\\
&\textstyle\sum \gamma_{11}^2\gamma_{12}^2 = +\ 1.4418, && \textstyle\sum \gamma_{11}^2\gamma_{22}^2 = +\ 8.2049\\
&\textstyle\sum \gamma_{11}\gamma_{12}\gamma_{21}\gamma_{22} = -\ 1.4418, && \textstyle\sum \gamma_{21}^2\gamma_{12}^2 = +\ 4.3255\\
&\textstyle\sum \gamma_{21}^2\gamma_{22}^2 = +\ 1.4418,
\end{aligned}
\tag{76}
$$

The values of $N'_{200}(2)$ and $N'_{020}(2)$ may be computed from the two first equations (70), and the three moments $N'_{200}(2)$, $N'_{020}(2)$, and $N'_{002}(2)$ from the three equations (72). More accurate values, however, are obtained through conjoining these expressions, further increased by the two first expressions (74). Putting

$$
\begin{aligned}
a' &= \textstyle\sum \gamma_{11}^2\nu'_{200} + \sum \gamma_{12}^2\nu'_{020} + 2\sum \gamma_{11}\gamma_{12}\nu'_{110}\\
b' &= \textstyle\sum \gamma_{21}^2\nu'_{200} + \sum \gamma_{22}^2\nu'_{020} + 2\sum \gamma_{21}\gamma_{22}\nu'_{110}\\
c' &= \qquad\qquad \textstyle\sum \gamma_{32}^2\nu'_{020}
\end{aligned}
\tag{77}
$$

which give

$$a'+b'+c' = \sum \nu'_{200} + \sum \nu'_{020}$$

We obtain

$$
\begin{aligned}
a' &= \vartheta_2 N'_{200}(2)\textstyle\sum (1-\gamma_{13}^2)^2 + \vartheta_2 N'_{020}(2)\sum \gamma_{13}^2\gamma_{23}^2 + \vartheta_2 N'_{002}(2)\sum \gamma_{12}^2\gamma_{32}^2\\
b' &= \vartheta_2 N'_{200}(2)\textstyle\sum \gamma_{13}^2\gamma_{23}^2 + \vartheta_2 N'_{020}(2)\sum (1-\gamma_{23}^2)^2 + \vartheta_2 N'_{002}(2)\sum \gamma_{22}^2\gamma_{32}^2\\
c' &= \vartheta_2 N'_{200}(2)\textstyle\sum \gamma_{32}^2\gamma_{12}^2 + \vartheta_2 N'_{020}(2)\sum \gamma_{32}^2\gamma_{22}^2 + \vartheta_2 N'_{002}(2)\sum (1-\gamma_{33}^2)^2
\end{aligned}
\tag{78}
$$

We may observe that

$$
\begin{aligned}
\gamma_{12}^2\gamma_{32}^2 &= (\gamma_{12}\gamma_{32}+\gamma_{11}\gamma_{31})^2 = \gamma_{13}^2\gamma_{33}^2\\
\gamma_{22}^2\gamma_{32}^2 &= (\gamma_{22}\gamma_{32}+\gamma_{21}\gamma_{31})^2 = \gamma_{23}^2\gamma_{33}^2
\end{aligned}
$$

so that the coefficients in (78) depend only on the three direction cosines γ_{13}, γ_{23}, and γ_{33}, occurring in the derivation of the velocity ellipsoid from the radial velocities.

Substituting in (78) the numerical values of the power sums of the coefficients γ_{ij}, we get the following normal equations:

$$
\begin{aligned}
25.3070\,\vartheta_2 N'_{200}(2) + 3.2574\,\vartheta_2 N'_{020}(2) + 3.2030\,\vartheta_2 N'_{002}(2) &= a' \\
3.2574\,\vartheta_2 N'_{200}(2) + 25.0694\,\vartheta_2 N'_{020}(2) + 3.3200\,\vartheta_2 N'_{002}(2) &= b' \\
3.2030\,\vartheta_2 N'_{200}(2) + 3.3200\,\vartheta_2 N'_{020}(2) + 26.0629\,\vartheta_2 N'_{002}(2) &= c'
\end{aligned} \tag{79}
$$

which equations may be directly solved for the unknown quantities, $\vartheta_2 N'_{200}(2)$, $\vartheta_2 N'_{020}(2)$, and $\vartheta_2 N'_{002}(2)$. We then get

$$
\begin{aligned}
\vartheta_2 N'_{200}(2) &= +0.040677\,a' - 0.004703\,b' - 0.004400\,c', \\
\vartheta_2 N'_{020}(2) &= -0.004703\,a' + 0.041117\,b' - 0.004660\,c', \\
\vartheta_2 N'_{002}(2) &= -0.004400\,a' - 0.004660\,b' + 0.039503\,c'.
\end{aligned} \tag{80}
$$

The values of a', b', and c' are given in (77) as linear functions of the moments of the proper motions. Hence the equations (80) give the moments N'_{200}, N'_{020}, and N'_{002} in the system of coördinates K_2 multiplied by ϑ_2.

45. It remains to compute the moments $N'_{011}(2)$, $N'_{101}(2)$, and $N'_{110}(2)$.

Putting

$$
\begin{aligned}
d' &= \sum \gamma_{22}\gamma_{32}\nu'_{020} + \sum \gamma_{21}\gamma_{32}\nu'_{110} \\
e' &= \sum \gamma_{32}\gamma_{12}\nu'_{020} + \sum \gamma_{11}\gamma_{32}\nu'_{110} \\
f' &= \sum \gamma_{12}\gamma_{22}\nu'_{020} + \sum (\gamma_{11}\gamma_{22} + \gamma_{21}\gamma_{12})\nu'_{110} + \sum \gamma_{11}\gamma_{21}\nu'_{200}
\end{aligned} \tag{81}
$$

we obtain

$$
\begin{aligned}
\vartheta_2 N'_{011}(2) \sum (2\gamma_{22}^2\gamma_{32}^2 + \gamma_{21}^2\gamma_{32}^2) &= d' \\
\vartheta_2 N'_{101}(2) \sum (2\gamma_{32}^2\gamma_{12}^2 + \gamma_{11}^2\gamma_{32}^2) &= e' \\
\vartheta_2 N'_{110}(2) \sum (2\gamma_{11}^2\gamma_{21}^2 + 2\gamma_{12}^2\gamma_{22}^2 + (\gamma_{11}\gamma_{22} + \gamma_{21}\gamma_{12})^2) &= f'
\end{aligned} \tag{82}
$$

But according to the relations between the direction cosines γ_{ij}, we get, after some rather long reductions,

$$
\begin{aligned}
2\gamma_{22}^2\gamma_{32}^2 + \gamma_{21}^2\gamma_{32}^2 &= \gamma_{13}^2 + 2\gamma_{23}^2\gamma_{33}^2 \\
2\gamma_{32}^2\gamma_{12}^2 + \gamma_{11}^2\gamma_{32}^2 &= \gamma_{23}^2 + 2\gamma_{33}^2\gamma_{13}^2 \\
2\gamma_{11}^2\gamma_{21}^2 + 2\gamma_{12}^2\gamma_{22}^2 + (\gamma_{11}\gamma_{22} + \gamma_{21}\gamma_{12})^2 &= \gamma_{33}^2 + 2\gamma_{13}^2\gamma_{23}^2
\end{aligned}
$$

so that the expressions for $N'_{011}(2)$, $N'_{101}(2)$, and $N'_{110}(2)$ are:

$$
\begin{aligned}
\vartheta_2 N'_{011}(2) \sum (\gamma_{13}^2 + 2\gamma_{23}^2\gamma_{33}^2) &= d' \\
\vartheta_2 N'_{101}(2) \sum (\gamma_{23}^2 + 2\gamma_{33}^2\gamma_{13}^2) &= e' \\
\vartheta_2 N'_{110}(2) \sum (\gamma_{33}^2 + 2\gamma_{13}^2\gamma_{23}^2) &= f'
\end{aligned}
$$

The sums of the powers of γ_{ij} have the following values:

$$\begin{aligned} \sum \gamma_{13}^2 &= +16.2326, & \sum \gamma_{23}^2\gamma_{33}^2 &= +3.3200 \\ \sum \gamma_{23}^2 &= +16.3532, & \sum \gamma_{33}^2\gamma_{13}^2 &= +3.2030 \\ \sum \gamma_{33}^2 &= +15.4141, & \sum \gamma_{13}^2\gamma_{23}^2 &= +3.2574 \end{aligned} \tag{83}$$

so that the equations take the form:

$$\begin{aligned} 22.8726\ \vartheta_2 N'_{011}(2) &= d' \\ 22.7593\ \vartheta_2 N'_{101}(2) &= e' \\ 21.9289\ \vartheta_2 N'_{110}(2) &= f' \end{aligned} \tag{84}$$

The formulae (80) and (84) give us the values of all moments of the second order in the system K_2. These moments may, according to the formulae in §30, be reduced to moments about the means, which by the methods in chapter III give us the characteristics of the velocity ellipsoid.

46. *Numerical applications.*—The determination of the velocity ellipsoid from the proper motions is based on the moments of the second order of these motions. These moments are given in the following table for each one of the 48 squares.

With the help of this table and the values of the direction cosines γ_{ij} given in table 9, we get the following values of the quantities a', b', c', d', e', f',*

$$\begin{aligned} a' &= +6.3312 & d' &= +0.7506 \\ b' &= +3.5250 & e' &= +0.3613 \\ c' &= +2.5094 & f' &= -0.1751 \end{aligned}$$

Substituting these values in (80) and (84) we get

$$\begin{aligned} \vartheta_2 N'_{200}(2) &= +0.2395 & \vartheta_2 N'_{011}(2) &= +0.0257 \\ \vartheta_2 N'_{020}(2) &= +0.0939 & \vartheta_2 N'_{101}(2) &= +0.0122 \\ \vartheta_2 N'_{002}(2) &= +0.0547 & \vartheta_2 N'_{110}(2) &= -0.0051 \end{aligned} \tag{85}$$

To determine ϑ_2 we have to compare these values with the values of N'_{200}, N'_{020}, etc., that have been obtained in §40 from the radial velocities. The "mixed" moments $N'_{011}(2)$, $N'_{101}(2)$, and $N'_{110}(2)$ are not appropriate for this purpose. And even $N'_{200}(2)$, $N'_{020}(2)$, and $N'_{002}(2)$, separately treated, lead to somewhat deviating, though usable, values. It is better to use the *product* of these three values which is, approximately, proportional to the square of the volume of the velocity ellipsoid.

We get from the radial velocities

$$N'_{200}(2) \times N'_{020}(2) \times N'_{002}(2) = 4120.38$$

* These numerical values are actually obtained from formulae that deviate somewhat from the definitions given in (77).

TABLE 11

Moments of the second order of the proper motions of all stars (4182) brighter than the sixth magnitude

Sq.	N.	ν'_{200}	ν'_{110}	ν'_{020}	Sq.	N.	ν'_{200}	ν'_{110}	ν'_{020}
GA_1	49	+0.01418	−0.00204	+0.00322	GD_1	90	+0.00417	−0.00042	+0.00544
GA_2	62	+0.01932	−0.00129	+0.00310	GD_2	79	+0.00418	−0.00101	+0.00180
					GD_3	101	+0.00498	+0.00010	+0.00240
$m.$	(111)	+0.01675	−0.00166	+0.00316	GD_4	107	+0.01124	−0.00161	+0.00255
					GD_5	115	+0.00637	+0.00026	+0.00122
Σ	111	+0.03350	−0.00333	+0.00632	GD_6	170	+0.00247	+0.00006	+0.00168
					GD_7	144	+0.00238	−0.00026	+0.00116
GB_1	80	+0.00744	+0.00203	+0.00662	GD_8	166	+0.00476	−0.00024	+0.00107
GB_2	73	+0.00986	+0.00318	+0.00456	GD_9	160	+0.00700	+0.00040	+0.00203
GB_3	72	+0.00802	+0.00132	+0.00177	GD_{10}	106	+0.00826	+0.00120	+0.00234
GB_4	73	+0.00654	+0.00312	+0.00579	GD_{11}	106	+0.00582	+0.00295	+0.00401
GB_5	55	+0.00354	+0.00227	+0.00868	GD_{12}	97	+0.00438	+0.00132	+0.00356
GB_6	69	+0.00398	+0.00344	+0.00794					
GB_7	57	+0.01285	+0.00316	+0.00938	$m.$		+0.00550	+0.00023	+0.00244
GB_8	54	+0.00773	−0.00042	+0.00273					
GB_9	52	+0.00866	−0.00091	+0.00640	Σ	1441	+0.06601	+0.00275	+0.02926
GB_{10}	78	+0.00487	−0.00164	+0.00452					
					GE_1	67	+0.00593	−0.00321	+0.00713
$m.$		+0.00735	+0.00021	+0.00584	GE_2	62	+0.01081	−0.00391	+0.00524
					GE_3	61	+0.01020	−0.00033	+0.00623
Σ	663	+0.07349	+0.00213	+0.05839	GE_4	72	+0.01212	+0.00219	+0.00389
					GE_5	74	+0.00710	+0.00122	+0.00544
GC_1	94	+0.00391	+0.00072	+0.00154	GE_6	75	+0.00983	−0.00303	+0.00663
GC_2	135	+0.00600	+0.00009	+0.00106	GE_7	60	+0.01042	−0.00112	+0.00254
GC_3	98	+0.00526	+0.00002	+0.00161	GE_8	75	+0.01417	+0.00017	+0.00323
GC_4	95	+0.00395	−0.00026	+0.00053	GE_9	60	+0.01042	+0.00229	+0.00508
GC_5	78	+0.00580	−0.00167	+0.00180	GE_{10}	81	+0.00444	+0.00139	+0.00796
GC_6	92	+0.00478	−0.00277	+0.00416					
GC_7	84	+0.00339	+0.00018	+0.00190	$m.$		+0.00954	−0.00043	+0.00534
GC_8	78	+0.00368	+0.00157	+0.00176					
GC_9	102	+0.00843	+0.00044	+0.00167	Σ	687	+0.09544	−0.00434	+0.05337
GC_{10}	120	+0.01125	+0.00131	+0.00188					
GC_{11}	121	+0.00339	+0.00004	+0.00118	GF_1	52	+0.00678	−0.00058	+0.00366
GC_{12}	71	+0.00490	+0.00081	+0.00570	GF_2	60	+0.00667	+0.00092	+0.00371
$m.$		+0.00540	+0.00004	+0.00207	$m.$		+0.00672	+0.00017	+0.00368
Σ	1168	+0.06474	+0.00048	+0.02479	Σ	112	+0.01345	+0.00034	+0.00737
						$\Sigma\Sigma$	+0.34663	−0.00197	+0.17950

and from the proper motions

$$\vartheta_2 N'_{200}(2) \times \vartheta_2 N'_{020}(2) \times \vartheta_2 N'_{002}(2) = 0.00123465$$

which give the following value of ϑ_2:

$$\vartheta_2 = 0.006692.$$

Using this value of ϑ_2 we derive from (85) the following values of $N'_{200}(2)$, etc., from the proper motions

$$\begin{aligned} N'_{200}(2) &= +35.79, & N'_{011}(2) &= +3.840 \\ N'_{020}(2) &= +14.03, & N'_{101}(2) &= +1.823 \\ N'_{002}(2) &= +\ 8.20, & N'_{110}(2) &= -0.765 \end{aligned}$$

From these values we now derive the quantities $N_{ijk}(2)$ according to the formulae in §30.

$$\begin{aligned} N_{200}(2) &= +24.58, & N_{011}(2) &= +1.435 \\ N_{020}(2) &= +11.88, & N_{101}(2) &= -3.082 \\ N_{002}(2) &= +\ 6.06, & N_{110}(2) &= -6.261 \end{aligned}$$

The equation of Jacobi now takes the form

$$I(t) = -t^3 + 42.52t^2 - 462.27t + 1423.97 = 0$$

which has the roots

$$\begin{aligned} t_1 &= 5.518 = \sigma_1^2, \text{ so that } \sigma_1 = 2.349 \text{ sir.} \\ t_2 &= 9.323 = \sigma_2^2, \qquad\quad \sigma_2 = 3.053 \text{ sir.} \\ t_3 &= 27.684 = \sigma_3^2, \qquad\quad \sigma_3 = 5.262 \text{ sir.} \end{aligned}$$

The direction cosines of the axes are

$$\begin{aligned} l_1 &= +0.1274, & l_2 &= +0.3863, & l_3 &= +0.9136 \\ m_1 &= -0.0972, & m_2 &= +0.9215, & m_3 &= -0.3761 \\ n_1 &= +0.9871, & n_2 &= +0.0403, & n_3 &= -0.1552 \end{aligned}$$

giving the following directions of the three axes:

$$\begin{aligned} L_1 &= 322^\circ.64, & L_2 &= 67^\circ.26, & L_3 &= 337^\circ.62 \\ B_1 &= +80.79, & B_2 &= +2.31, & B_3 &= -8.92 \end{aligned}$$

It may be observed that the *absolute* values of σ_1, σ_2, and σ_3 depend only on the radial velocities and not on the proper motions. The *relative* values of the axes as well as their directions are, however, directly obtained from the proper motions.

47. In the preceding section we have derived the value of ϑ_2 by comparing the moments $N'_{200}(2)$, $N'_{020}(2)$, etc., obtained from the radial velocities with the values $\vartheta_2 N'_{200}(2)$, $\vartheta_2 N'_{020}(2)$, etc., obtained from the proper motions.

We can compute the value of ϑ_2 in another way. From the velocity ellipsoid obtained from the radial velocities, we may compute the moments

$$N'_{ijo}(3)$$

for each one of the 48 squares with the help of the formulae (66) in §43. Comparing these values with the values of ν'_{ijo} directly obtained from the proper motions we may compute the value of ϑ_2 for each square. The values obtained in this manner are given in §17. We found there that we get a different value of ϑ_2 in the squares *GC* and *GD* along the Milky Way and in the other squares.

From the *GC* and *GD* squares $\vartheta_2 = 0.0048$
From the *GA*, *GB*, *GE*, *GF* squares $\vartheta_2 = 0.0086$

We may use these results in the calculation of $a' : \vartheta_2$, $b' : \vartheta_2$, etc., in the formulae (80)–(84) treating separately the terms from galactic and non-galactic squares. The numerical values of $N'_{200}(2)$, $N'_{020}(2)$, etc., are in this manner somewhat changed. We get

$$N'_{200}(2) = +33.84, \quad N'_{011}(2) = +2.007$$
$$N'_{020}(2) = +13.92, \quad N'_{101}(2) = +4.029$$
$$N'_{002}(2) = +\ 9.39, \quad N'_{110}(2) = +0.012$$

and the moments about the mean:

$$N_{200}(2) = +22.48, \quad N_{011}(2) = -0.335$$
$$N_{020}(2) = +11.83, \quad N_{101}(2) = -1.429$$
$$N_{002}(2) = +\ 6.77, \quad N_{110}(2) = -4.864$$

We then get the values of σ_1, σ_2, and σ_3 and their directions as given in §18.

48. *Summary of formulae for computing Apex*

(*a*) *From the parallax stars.*—Given for each star

$$\pi,\ u,\ v,\ W.$$

First we compute the linear velocities U and V in the system K_3 by the formulae

$$U = u : \pi; \quad V = v : \pi.$$

We then compute for each star U'', V'', W'' in the system K_2 by the formulae

$$U'' = \gamma_{11}U + \gamma_{12}V + \gamma_{13}W$$
$$V'' = \gamma_{21}U + \gamma_{22}V + \gamma_{23}W$$
$$W'' = \gamma_{31}U + \gamma_{32}V + \gamma_{33}W$$

Taking the mean of U'', V'', W'' we now get the components of the stream velocity

$$U_o'' = \bar{U}'', \quad V''_o = \bar{V}'', \quad W_o'' = \bar{W}''$$

The solar velocity and the coördinates of the Apex are now found from the formulae

$$\Omega \cos B \cos L = -U_o''$$
$$\Omega \cos B \sin L = -V_o''$$
$$\Omega \sin B \qquad = -W_o''$$

(β) *From the radial velocities.*—Given for each star

$$W = \text{radial velocity.}$$

We first compute for each square the mean value of W

$$\overline{W} = N'_{001}(3)$$

Then we get the components $U_\circ''$, $V_\circ''$, $W_\circ''$ of the stream velocity from the formulae:

$$16.2324\ U_\circ'' = \sum \gamma_{13}\, \overline{W}$$
$$16.3552\ V_\circ'' = \sum \gamma_{23}\, \overline{W}$$
$$15.4132\ W_\circ'' = \sum \gamma_{33}\, \overline{W}$$

the summation being extended to all the 48 squares.

The values of $U_\circ''$, $V_\circ''$, $W_\circ''$ being thus obtained we get Ω, B, L as for the parallax stars.

(γ) *From the proper motions.*—Given for each star

$$u,\ v.$$

We first compute for each square the mean values of u and v

$$\bar{u},\ \bar{v}.$$

Then we get the components $U_\circ''$, $V_\circ''$, $W_\circ''$ of the stream-velocity, multiplied by ϑ_1 from the formulae:

$$31.7674\ \vartheta_1 U_\circ'' = \sum \gamma_{11}\bar{u} + \sum \gamma_{12}\bar{v}$$
$$31.6468\ \vartheta_1 V_\circ'' = \sum \gamma_{21}\bar{u} + \sum \gamma_{22}\bar{v}$$
$$32.5859\ \vartheta_1 W_\circ'' = \sum \gamma_{33}\bar{v}$$

Using now the expressions for $U_\circ''$, $V_\circ''$, $W_\circ''$ given in (α) we get the values of L and B.

From the foregoing formulae we get the value of

$$\vartheta_1 \sqrt{U_\circ''^2 + V_\circ''^2 + W_\circ''^2} = \vartheta_1 \Omega$$

and using the value of Ω obtained from the radial velocities we thus get the values of ϑ_1, and hence the mean parallax of the stars.

49. *Summary of formulae for computing the velocity ellipsoid.*

(α) *From the parallax stars.*—The values of U'', V'', W'' are computed as in §48 for each star.

Then the moments N_{ijk} of the second order are computed according to the formulae of mathematical statistics and the equation of Jacobi is found

$$G(t)=\begin{vmatrix} N_{200}-t, & N_{110}, & N_{101} \\ N_{110}, & N_{020}-t, & N_{011} \\ N_{101}, & N_{011}, & N_{002}-t \end{vmatrix}=0$$

The three roots of this equation are equal to the square of the three axes of the velocity ellipsoid.

The direction cosines l, m, n, of this axis are obtained from

$$\frac{l}{G_{11}}=\frac{m}{G_{12}}=\frac{n}{G_{13}}=\frac{1}{\sqrt{G_{11}^2+G_{12}^2+G_{13}^2}},$$

where G_{11}, G_{12}, G_{13}, are the subdeterminants of $G(t)$ corresponding to each one of the three roots.

(β) *From the radial velocities.*—First the dispersion σ from the radial velocities is computed for each square.

Then the quantities a, b, c, d, e, f, are computed according to the relations

$$a=\sum\gamma_{13}^2\sigma^2, \qquad d=\sum\gamma_{23}\gamma_{33}\sigma^2$$
$$b=\sum\gamma_{23}^2\sigma^2, \qquad e=\sum\gamma_{33}\gamma_{13}\sigma^2$$
$$c=\sum\gamma_{33}^2\sigma^2, \qquad f=\sum\gamma_{13}\gamma_{23}\sigma^2$$

Then the moments in the system K_2 are obtained from the formulae

$$N_{200}(2)=+0.123113\,a-0.029719\,b-0.033260\,c$$
$$N_{020}(2)=-0.029719\,a+0.124299\,b-0.035725\,c$$
$$N_{002}(2)=-0.033260\,a-0.035725\,b+0.137821\,c$$
$$6.6416N_{011}(2)=d,$$
$$6.4064N_{101}(2)=e,$$
$$6.5152N_{110}(2)=f.$$

The equation of Jacobi and the determination of the axes are now performed as in (α).

(γ) *From the proper motions.*—From the proper motions (u, v) of the stars we first compute the moments of the second order

$$\nu'_{200},\ \nu'_{020},\ \text{and}\ \nu'_{110}.$$

Then the quantities a', b', c', d', e', f' are computed according to the relations

$$a'=\sum\gamma_{11}^2\nu'_{200}+\sum\gamma_{12}^2\nu'_{020}+2\sum\gamma_{11}\gamma_{12}\nu'_{110}$$
$$b'=\sum\gamma_{21}^2\nu'_{200}+\sum\gamma_{22}^2\nu'_{020}+2\sum\gamma_{21}\gamma_{22}\nu'_{110}$$
$$c'=\sum\gamma_{32}^2\nu'_{020}$$

$$d' = \sum \gamma_{22}\gamma_{32}\nu'_{020} + \sum \gamma_{21}\gamma_{32}\nu'_{110}$$
$$e' = \sum \gamma_{32}\gamma_{12}\nu'_{020} + \sum \gamma_{11}\gamma_{32}\nu'_{110}$$
$$f' = \sum \gamma_{12}\gamma_{22}\nu'_{020} + \sum (\gamma_{11}\gamma_{22}+\gamma_{21}\gamma_{12})\nu'_{110} + \sum \gamma_{11}\gamma_{21}\nu'_{200}$$

The moments N'_{ijk} in the system K_2 are then computed from the formulae

$$\vartheta_2 N'_{200}(2) = +0.040677\ a' - 0.004703\ b' - 0.004400\ c'$$
$$\vartheta_2 N'_{020}(2) = -0.004703\ a' + 0.041117\ b' - 0.004660\ c'$$
$$\vartheta_2 N'_{002}(2) = -0.004400\ a' - 0.004660\ b' + 0.039503\ c'$$
$$22.8726\ \vartheta_2 N'_{011}(2) = d'$$
$$22.7593\ \vartheta_2 N'_{101}(2) = e'$$
$$21.9289\ \vartheta_2 N'_{110}(2) = f'$$

The value of ϑ_2 is determined as shown in §47. From $N'_{ijk}(2)$ we compute $N_{ijk}(2)$ form the equation of Jacobi, and continue as before.

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50. *Summary of numerical results.*—In table 12 I give a summary of the numerical results regarding the velocity of the sun, the Apex, and the velocity ellipsoid as obtained from parallax stars, from radial velocity stars, and from proper motions.

In the second line is given the number of stars belonging to the different groups. The third, fourth, and fifth lines give the components of the stream velocity. The values in the last column, from the proper motion stars, are obtained with the use of the value 4.009 Sir/st. for the velocity of the sun found from the radial velocities. In like manner the values of σ_1, σ_2, and σ_3 are obtained from the proper motion stars with the help of the value of ϑ_2 found from the radial velocities.

F. Dissection of the Frequency Distribution

51. We have found in the preceding chapters that the distribution of the velocities of the stars is nearly represented through a velocity ellipsoid having its longest axis in the Milky Way and the two shorter axes of nearly the same size.

If, however, the characteristics of the third and fourth order are taken into consideration, we find that even these have noteworthy values and hence it is necessary to find out whether or not these deviations in the frequency distribution are of such a nature that a dissection of the frequency function into simpler components is possible. We must particularly examine the suggestion of Kapteyn in the year 1904, which preceded the ellipsoidal theory of Schwarzschild regarding the representation of the velocity distribution of the stars through two Maxwellian streams, to determine whether it is compatible with the observations or not.

With Wicksell, I have, in *Meddelanden* (103), examined the general problem of a frequency distribution into two *ellipsoidal* streams. In *Meddelanden* (II, 9)

TABLE 12

Velocity of the sun, apex and the velocity-ellipsoid from different groups of stars

	Parallax stars	Radial velocity stars	Proper motion stars
N	646	1986	4182
U_o''	− 3.201 Sir/st	− 3.371 Sir/st	− 3.350 Sir/st
V_o''	− 1.385 "	− 1.447 "	− 1.643 Sir/st
W_o''	− 1.413 "	− 1.619 "	− 1.466 Sir/st
Ω	3.764 Sir/st	4.009 Sir/st	(4.009) Sir/st
	17.83 km/sec	18.99 km/sec	(18.99) km/sec
L	23°.40	22°.87	26°.12
B	+ 22.06	+ 23°.82	+ 21.45
σ_1	3.410 Sir/st	3.294 Sir/st	2.546 Sir/st
	16.15 km/sec	15.61 km/sec	12.06 km/sec
L_1	233°.06	270°.00	52°.86
B_1	+ 83°.07	+ 72°.77	+ 75°.84
σ_2	4.088 Sir/st	2.835 Sir/st	3.185 Sir/st
	19.37 km/sec	13.43 km/sec	15.09 km/sec
L_2	71°.36	69°.58	69°.96
B_2	+ 7°.65	+ 16.17	− 13°.40
σ_3	5.880 Sir/st	4.125 Sir/st	4.944 Sir/st
	27.86 km/sec	19.94 km/sec	23.42 km/sec
L_3	341°.29	341°.20	339°.01
B_3	+ 2.84	− 5°.71	− 3°.93

I have discussed the problem of Kapteyn with respect to two dimensions, which is sufficient when the distribution of the frequencies is examined for each individual square. I shall here extend this investigation to three variates, the three velocity components of the stars.

Let $F\ (x,\ y,\ z)$ be the given velocity distribution. Through established methods in mathematical statistics the moments and the characteristics of this function are supposed to be known.

Let φ_1 and φ_2 be the frequency functions of two Maxwellian streams so that

$$\varphi_1 = \frac{1}{a_1^3(2\pi)^{3/2}}\ e^{-\frac{(x-m_1)^2+(y-n_1)^2+(z-s_1)^2}{2a_1^2}}$$

$$\varphi_2 = \frac{1}{a_2^3(2\pi)^{3/2}}\ e^{-\frac{(x-m_2)^2+(y-n_2)^2+(z-s_2)^2}{2a_2^2}}$$

If N is the total number of stars considered and we suppose with Kapteyn that the observed frequency distribution F may be divided into two Maxwellian streams consisting of N_1 and N_2 stars, respectively, we have

$$N F = N_1\varphi_1 + N_2\varphi_2$$

or putting

(86) $$p = N_1/N, \quad q = N_2/N$$

(86*) $$F = p\varphi_1 + q\varphi_2.$$

We have to determine the characteristics of these components, i. e., the following 10 quantities:

(87) $$p,\ q;\ m_1,\ n_1,\ s_1;\ m_2,\ n_2,\ s_2;\ \alpha_1,\ \alpha_2$$

Hence we have to form ten independent relations between these quantities and the characteristics of the given frequency function F.

We denote the moments of F about the means (m, n, s) by N_{ijk} so that

(88) $$N_{ijk} = \int_{-\infty}^{+\infty}\!\!\int\!\!\int dx\, dy\, dz\, (x-m)^i\, (y-n)^j\, (z-s)^k . F.$$

If the moments of the two streams about the same means (m, n, s) are $\nu'_{ijk}(1)$ and $\nu'_{ijk}(2)$ we have

(89) $$\nu'_{ijk}(1) = \int_{-\infty}^{+\infty}\!\!\int\!\!\int dx\, dy\, dz\, (x-m)^i\, (y-n)^j\, (z-s)^k\, \varphi_1$$

$$\nu'_{ijk}(2) = \int_{-\infty}^{+\infty}\!\!\int\!\!\int dx\, dy\, dz\, (x-m)^i\, (y-n)^j\, (z-s)^k\, \varphi_2$$

If now the equation (86*) is multiplied by $(x-m)^i\, (y-n)^j\, (z-s)^k$ and integrated, we obtain

(90) $$N_{ijk} = p\nu'_{ijk}(1) + q\nu'_{ijk}(2)$$

Giving to i, j, k all possible positive values including zero, we obtain an arbitrary number of relations, of which, however, only ten—which may be chosen in different ways—may be considered as independent of one another. The other equations lead to an indefinite number of relations between the moments N_{ijk} that must be—more or less approximately—fulfilled, if a dissection into two Maxwellian streams is really possible.

We observe that

(91) $$N_{000} = \nu'_{000}(1) = \nu'_{000}(2) = 1$$

so that we get the following equation of the order zero:

(92) $$1 = p + q.$$

From the definition m, n, and s as the means of F we get

$$N_{100}=N_{010}=N_{001}=0$$

so that we obtain the following equations of the first order:

$$\begin{aligned}0&=p\nu'_{100}(1)+q\nu'_{100}(2)\\0&=p\nu'_{010}(1)+q\nu'_{010}(2)\\0&=p\nu'_{001}(1)+q\nu'_{001}(2)\end{aligned}$$

We have, however, as easily found,

$$\begin{aligned}\nu'_{100}(1)&=m_1-m\\\nu'_{010}(1)&=n_1-n\\\nu'_{001}(1)&=s_1-s\end{aligned}$$

so that the relations of the first order take the form

$$\begin{aligned}0&=p(m_1-m)+q(m_2-m)\\0&=p(n_1-n)+q(n_2-n)\\0&=p(s_1-s)+q(s_2-s)\end{aligned}\tag{93}$$

Combining these with (92) we obtain

$$\begin{aligned}m_1-m&=q(m_1-m_2), & m_2-m&=-p(m_1-m_2)\\n_1-n&=q(n_1-n_2), & n_2-n&=-p(n_1-n_2)\\s_1-s&=q(s_1-s_2), & s_2-s&=-p(s_1-s_2)\end{aligned}\tag{94}$$

We have six relations of the second order which have the form

$$\begin{aligned}N_{200}&=p\nu'_{200}(1)+q\nu'_{200}(2)\\N_{011}&=p\nu'_{011}(1)+q\nu'_{011}(2)\end{aligned}$$

The four remaining relations are obtained through permutation of the indices.

We have, however,

$$\begin{aligned}\nu'_{200}(1)&=a_1^2+(m_1-m)^2\\\nu'_{011}(1)&=(n_1-n)\,(s_1-s)\end{aligned}$$

or substituting the values of m_1-m, n_1-n and s_1-s from (94)

$$\begin{aligned}\nu'_{200}(1)&=a_1^2+q^2(m_1-m_2)^2\\\nu'_{011}(1)&=q^2(n_1-n_2)\,(s_1-s_2)\end{aligned}$$

so that the six relations of the second order are

$$\begin{aligned}N_{200}&=pa_1^2+qa_2^2+pq(m_1-m_2)^2\\N_{020}&=pa_1^2+qa_2^2+pq(n_1-n_2)^2\\N_{002}&=pa_1^2+qa_2^2+pq(s_1-s_2)^2\\N_{011}&=pq(n_1-n_2)\,(s_1-s_2)\\N_{101}&=pq(s_1-s_2)\,(m_1-m_2)\\N_{110}&=pq(m_1-m_2)\,(n_1-n_2)\end{aligned}\tag{95}$$

From the last three equations it follows that the quotients $(m_1-m_2):(n_1-n_2):(s_1-s_2)$ are directly known as function of the moments N_{011}, N_{101}, N_{110}. We have, indeed,

$$\frac{N_{101}}{N_{110}}=\frac{s_1-s_2}{n_1-n_2}$$
$$\frac{N_{110}}{N_{011}}=\frac{m_1-m_2}{s_1-s_2}$$
$$\frac{N_{011}}{N_{101}}=\frac{n_1-n_2}{m_1-m_2}$$

52. For maintaining the symmetrical form of the equations we put

$$\begin{aligned} m_1-m_2&=a\,D \\ n_1-n_2&=b\,D \\ s_1-s_2&=c\,D \end{aligned} \tag{96}$$

where

$$D=\sqrt{(m_1-m_2)^2+(n_1-n_2)^2+(s_1-s_2)^2} \tag{97}$$

and hence is equal to the distance between the centers of the two components.

The quantities a, b, c are equal to the direction cosines of D and we have

$$a^2+b^2+c^2=1 \tag{98}$$

Our equations of the second order now take the form

$$\begin{aligned} N_{200}&=pa_1^2+qa_2^2+pq\,a^2D^2 \\ N_{020}&=pa_1^2+qa_2^2+pq\,b^2D^2 \\ N_{002}&=pa_1^2+qa_2^2+pq\,c^2D^2 \\ N_{011}&=\qquad\qquad pq\,bc\,D^2 \\ N_{101}&=\qquad\qquad pq\,ca\,D^2 \\ N_{110}&=\qquad\qquad pq\,ab\,D^2 \end{aligned} \tag{99}$$

Hence we have

$$a\,N_{011}=b\,N_{101}=c\,N_{110} \tag{100}$$

from which equations a, b, and c are determined.

From (99) we obtain

$$\begin{aligned} \frac{N_{020}-N_{002}}{N_{011}}&=\frac{N_{110}}{N_{101}}-\frac{N_{101}}{N_{110}}=\frac{b}{c}-\frac{c}{b} \\ \frac{N_{002}-N_{200}}{N_{101}}&=\frac{N_{011}}{N_{110}}-\frac{N_{110}}{N_{011}}=\frac{c}{a}-\frac{a}{c} \\ \frac{N_{200}-N_{020}}{N_{110}}&=\frac{N_{101}}{N_{011}}-\frac{N_{011}}{N_{101}}=\frac{a}{b}-\frac{b}{a} \end{aligned} \tag{101}$$

which are three relations between the moments of the second order that may be fulfilled if a dissection into two Maxwellian streams is possible. The third of these relations is a direct consequence of the other two, so that only two independent relations exist here. The six equations of the second order give us four independent equations for the solution of our problem.

The equations of the third order are of the following types:

$$N_{300} = p\nu'_{300}(1) + q\nu'_{300}(2)$$
$$N_{021} = p\nu'_{021}(1) + q\nu'_{021}(2)$$

But
$$\begin{aligned}\nu'_{300}(1) &= 3(m_1 - m)a_1^2 + (m_1 - m)^3 \\ &= 3q(m_1 - m_2)a_1^2 + q^3(m_1 - m_2)^3 \\ &= 3qaDa_1^2 + q^3a^3D^3\end{aligned}$$

and
$$\begin{aligned}\nu'_{021}(1) &= (s_1 - s)a_1^2 + (n_1 - n)^2(s_1 - s_2) \\ &= q(s_1 - s_2)a_1^2 + q^3(n_1 - n_2)^2(s_1 - s) \\ &= qcDa_1^2 + q^3b^2cD^3\end{aligned}$$

and
$$\nu'_{300}(2) = -3paDa_2^2 - p^3a^3D^3$$
$$\nu'_{021}(2) = -pcDa_2^2 - p^3b^2cD^3$$

so that

(102)
$$N_{300} = 3pqaD(a_1^2 - a_2^2) - pq(p-q)a^3D^3$$
$$N_{021} = pqcD(a_1^2 - a_2^2) - pq(p-q)b^2cD^3$$

Similar expressions are obtained for other moments of the third order. The moment N_{111} gives the equation

(102*)
$$N_{111} = -abc\ pq\ (p-q)\ D^3,$$

so that all moments of the third order are expressed as linear functions of two quantities

$$pq\ D\ (a_1^2 - a_2^2) \quad \text{and} \quad pq\ (p-q)\ D^3,$$

depending upon the characteristics of the two components.

53. In similar manner we find that the moments of the fourth order are expressed as linear moments of three such quantities. Putting now

(103)
$$\begin{aligned} x_2 &= pa_1^2 + qa_2^2, & y_2 &= pq\ D^2 \\ x_3 &= pq\ D\ (a_1^2 - a_2^2), & y_3 &= pq\ (p-q)\ D^3 \\ x_4 &= pq\ (a_1^2 - a_2^2)^2. & y_4 &= pq\ (p-q)\ (a_1^2 - a_2^2)\ D^2 \\ & & z_4 &= pq\ (1 - 6pq)\ D^4 \end{aligned}$$

we obtain the following expressions for determining the unknown characteristics of the components:

(104)
$$\begin{aligned} N_{200} &= x_2 + a^2y_2 \\ N_{020} &= x_2 + b^2y_2 \\ N_{002} &= x_2 + c^2y_2 \\ N_{011} &= bc\ y_2 \\ N_{101} &= ca\ y_2 \\ N_{110} &= ab\ y_2 \end{aligned}$$

$$
\begin{aligned}
N_{300} &= 3a\,x_3 - a^3\,y_3 \\
N_{030} &= 3b\,x_3 - b^3\,y_3 \\
N_{003} &= 3c\,x_3 - c^3\,y_3 \\
N_{021} &= c\,x_3 - b^2c\,y_3 \\
N_{012} &= b\,x_3 - bc^2\,y_3 \\
N_{201} &= c\,x_3 - a^2c\,y_3 \\
N_{102} &= a\,x_3 - ac^2\,y_3 \\
N_{210} &= b\,x_3 - ba^2\,y_3 \\
N_{120} &= a\,x_3 - b^2a\,y_3 \\
N_{111} &= -abc\,y_3
\end{aligned} \tag{105}
$$

In treating the moments of the fourth order it is convenient to use the characteristics of this order instead of the moments. To this end we use

$$
\begin{aligned}
K_{400} &= N_{400} - 3N_{200}^2 \\
K_{040} &= N_{040} - 3N_{020}^2 \\
K_{004} &= N_{004} - 3N_{002}^2
\end{aligned}
$$

and similar expressions for the other twelve characteristics of this order, which are defined in *L. M.* 103. For our present purpose, we may limit ourselves to the three expressions given above. For these we obtain the relations:

$$
\begin{aligned}
K_{400} &= 3x_4 - 6a^2y_4 + a^4z_4 \\
K_{040} &= 3x_4 - 6b^2y_4 + b^4z_4 \\
K_{004} &= 3x_4 - 6c^2y_4 + c^4z_4
\end{aligned} \tag{106}
$$

54. The six relations (104) of the second order give us the values of a, b, c and of x_2 and y_2. Moreover, we obtain from them two relations between the moments of the second order which must be fulfilled if it is possible to dissect F into two Maxwellian components.

The ten relations (105) of the third order give us the values of x_3 and y_3 and as well as eight relations between the moments of F of the third order.

The fifteen relations of the fourth order give us the values of x_4, y_4, and z_4 besides twelve relations between the moments of F.

55. For determining our ten unknown quantities (87) we may use

1 relation of the 0th order
3 relations of the 1st order
4 relations of the 2d order
and 2 relations of the 3d order

The order of the computation is then the following.

From the four relations of the second order supplied by the relation $a^2+b^2+c^2=1$, we obtain a, b, c, x_2 and y_2.

From two relations of the third order we get the values of x_3 and y_3.

From the definitions (103) of y_2 and y_3 it follows that

$$D = y_3 : y_2\,(p-q) \tag{107}$$

which value of D substituted in y_2 gives

$$y_2 = pq\, y_3^2 : y_2^2\,(p-q)^2$$

so that

$$\frac{(p-q)^2}{pq} = \frac{y_3^2}{y_2^3}$$

or

(108)
$$\sqrt{\frac{p}{q}} - \sqrt{\frac{q}{p}} = \frac{y_3}{y_2^{3/2}}$$

which equation gives the value of $\sqrt{p} : \sqrt{q}$, which combined with $p+q=1$ gives p and q.

From (107) we now get the value of D and from

(109)
$$\begin{aligned} a_1^2 - a_2^2 &= x_3 : pq\, D \\ p a_1^2 + q a_2^2 &= x_2 \end{aligned}$$

the values of the dispersions a_1 and a_2.

From (96) we now obtain $m_1 - m_2$, $n_1 - n_2$, and $s_1 - s_2$ and from (94) the values of

$$m_1,\ n_1,\ s_1;\quad m_2,\ n_2,\ s_2$$

and the determination of the characteristics is completed.

At the determination of x_2 and y_2 from the moments of the second order we may eventually treat the equations as relations obtained according to the method of least squares and then obtain the normal equations:

(110)
$$\begin{aligned} 3x_2 + y_2 &= N_{200} + N_{020} + N_{002} \\ x_2 + (a^4+b^4+c^4)y_2 &= a^2 N_{200} + b^2 N_{020} + c^2 N_{002} \end{aligned}$$

if only the moments N_{200}, N_{020}, N_{002} are taken into consideration.

In like manner we obtain from the moments of the third order

(111)
$$\begin{aligned} 3x_3 - (a^4+b^4+c^4)y_3 &= a\, N_{300} + b\, N_{030} + c\, N_{003} \\ 3(a^4+b^4+c^4)x_3 - (a^6+b^6+c^6)y_3 &= a^3 N_{300} + b^3 N_{030} + c^3 N_{003} \end{aligned}$$

Eventually we may use here all moments of the third order, giving different weight to all mixed moments. We then get the normal equations

(112)
$$\begin{aligned} 15x_3 - 3y_3 = {} & 3aN_{300} + 3bN_{030} + 3cN_{003} \\ & + 3cN_{021} + 3bN_{012} + 3cN_{201} \\ & + 3aN_{201} + 3bN_{210} + 3aN_{120} \\ 3x_3 - y_3 = {} & a^3 N_{300} + b^3 N_{030} + c^3 N_{003} \\ & + 3b^2 cN_{021} + 3bc^2 N_{012} + 3a^2 cN_{201} \\ & + 3ac^2 N_{102} + 3ba^2 N_{210} + 3b^2 aN_{120} \\ & + 6abcN_{111}. \end{aligned}$$

56. Instead of basing the dissection of the frequency function F only on moments to the third order we may also take the moments of the fourth order into consideration. This is generally even preferable as the excess is especially influenced by an addition of spherical components.

Determining x_4, y_4, and z_4 from (106) we may observe that

$$x_3^2 = x_4 y_2, \quad x_3 y_3 = y_2 y_4$$

and

$$\frac{x_4 z_4}{y_2 y_2} = (1-6\,pq)\,(\alpha_1^2-\alpha_2^2)^2$$

$$\frac{y_4 y_4}{y_2 y_2} = (p-q)^2\,(\alpha_1^2-\alpha_2^2)^2$$

so that

$$\frac{x_4 z_4}{y_4 y_4} = \frac{1-6\,pq}{(p-q)^2} = \frac{1-6\,pq}{1-4\,pq}$$

Putting now

(113) $$\kappa = x_4 z_4 : y_4 y_4$$

we get

(114) $$pq = \frac{1-\kappa}{6-4\kappa}$$

which equation combined with $p+q=1$ gives the values of p and q.

We may observe that for obtaining real values of p and q it is necessary that

$$\kappa < 1.$$

Having found p and q the other characteristics may be easily found from x_2, y_2, x_3, y_3, or by using the moments of the fourth order also.

57. *Summary.*—Using the numerical values of K_{400}, K_{040}, and K_{004} we get from (106) the values of x_4, y_4, and z_4.

From these values we compute

$$\kappa = \frac{x_4 z_4}{y_4 y_4}$$

and get the values of p and q as roots of the equation

$$t^2 - t + \frac{1-\kappa}{6-4\kappa} = 0$$

We now get the value of $\alpha_1^2 - \alpha_2^2$ from

(115) $$(\alpha_1^2-\alpha_2^2)^2 = x_4 : pq$$

We observe that this equation gives two values of $\alpha_1^2 - \alpha_2^2$ both of which must be examined.

We now get the value of D either from

(116) $$D^2 = \frac{y_4}{pq(p-q)\,(\alpha_1^2-\alpha_2^2)}$$

or from

(117) $$D^2 = y_2 : pq$$

Finally we get the values of α_1 and α_2 from

$$\alpha_1^2-\alpha_2^2 = \pm\sqrt{\frac{x_4}{pq}} \text{ and } x_2 = p\alpha_1^2+q\alpha_1^2.$$

The values of m_1, n_1, s_1; m_2, n_2, s_2 are then obtained from the moments of the first order.

We therefore find that it is possible to dissect a frequency distribution using only the following moments:

(118) $$\nu_{400},\ \nu_{040},\ \nu_{004}$$
$$\nu_{200},\ \nu_{020},\ \nu_{002}$$

supposing, however, that the direction cosines a, b, c are known. For determining these we must use the correlation moments

$$\nu_{011},\ \nu_{101},\ \nu_{110}$$

The moments (118) are all obtained from a direct treatment of the frequency curves of U, V, W separately, which is rather easy work to calculate.

58. If $\alpha_1 = \alpha_2$ we get

$$x_4 = y_4 = 0$$

and

$$K_{400} = a^4 z_4 = a^4 pq(1-6\,pq)\,D^4$$

If $pq > 1/6$, which implies that both components contain more than 21 per cent of the stars, the right hand member must be a negative quantity. The excess (E) in the frequency distribution of U is, however, defined by

$$E = \frac{K_{400}}{8\nu^2_{200}} = \frac{a^4 z_4}{8\nu^2_{200}}$$

(and similar expressions for the excess in V and W) so that each frequency distribution which may be dissected into two components, having the same dispersion, must possess a negative excess, as soon as the two components contain at least 21 per cent of the total number of stars.

59. Applying this conclusion to our 646 parallax stars, treated in Chapter III, we find from a direct computation that the actual excess in this distribution is positive, so that it is not possible to consider this group of stars as composed of two components with the same dispersion. It must be observed that these parallax stars contain only stars of the spectral types F, G, K, M, and that in such a selected group a dissection into two components may still be possible if the assumption $\alpha_1 = \alpha_2$ is abandoned.

PART II

THE DISTRIBUTION OF THE STARS

CHAPTER IV

MOVING CLUSTERS

60. Before treating the problem of the distribution of the stars in our Galaxy, we begin with a more simple question which gives, however, some hints regarding the treatment of the more general problem.

There are in the sky a great many, some hundreds, of more or less condensed groups of stars that may be considered either as temporary condensations in the Galaxy or as connected systems moving in parallel paths among the other stars. For some of the latter kind the proper motions may be determined. Such groups are called moving clusters, a denomination that may, perhaps, be extended to all, or a great part of the clusters. The best known of such moving clusters are the Pleiades, Praesepe, the Taurus cluster (the Hyades), and the Ursa Major cluster.

How may the distance and the velocity of such a cluster be determined?

We suppose the proper motions u and v and the radial velocity W of the stars, or of some of the stars, belonging to the cluster to be known. All these stars are supposed to describe parallel lines in the Galaxy.

Let U'', V'', W'', be the components of the velocity in this motion referred to a system of coördinates (K_2) fixed in the Galaxy, having the U'' V'' plane parallel to the celestial equator, the U'' axis being directed toward the equinoctial point.

The system of coördinates (K_3) used by the observer has the Z-axis (W) directed toward the cluster. We suppose it directed toward a point having the right ascension A and the declination D. The X-axis in this system may be parallel to the equator, the Y-axis directed toward the North Pole.

The components of the velocity of the cluster, when referred to this system K_3, may be denoted by

$$U,\ V,\ W.$$

Between the velocity components in K_2 and K_3, we then have the relations

$$\begin{aligned} U &= \gamma_{11}U'' + \gamma_{21}V'' + \gamma_{31}W'' \\ V &= \gamma_{12}U'' + \gamma_{22}V'' + \gamma_{32}W'' \\ W &= \gamma_{13}U'' + \gamma_{23}\gamma V'' + \gamma_{33}W'' \end{aligned} \tag{1}$$

where we easily find, from spherical trigonometry, that

$$\begin{aligned} \gamma_{11} &= \cos XX'' = -\sin A \\ \gamma_{21} &= \cos XY'' = \cos A \\ \gamma_{31} &= \cos XZ'' = 0 \\ \gamma_{12} &= \cos YX'' = -\sin D \cos A \\ \gamma_{22} &= \cos YY'' = -\sin D \sin A \\ \gamma_{32} &= \cos YZ'' = \cos D \\ \gamma_{13} &= \cos ZX'' = \cos \mathrm{D} \cos A \\ \gamma_{23} &= \cos ZY'' = \cos D \sin A \\ \gamma_{33} &= \cos ZZ'' = \sin D \end{aligned}$$

Of the components U, V, W, only W (the radial velocity) may be directly determined by the observer. The components U and V are not known but, instead of them, the proper motions u and v, which are related to the linear velocities U and V by the formulae

$$U=r\,u, \quad V=r\,v,$$

so that the equations (1) take the form

$$\begin{aligned} r\,u &= \gamma_{11}U''+\gamma_{21}V''+\gamma_{31}W'' \\ r\,v &= \gamma_{12}U''+\gamma_{22}V''+\gamma_{32}W'' \\ W &= \gamma_{13}U''+\gamma_{23}V''+\gamma_{33}W'' \end{aligned} \tag{2}$$

For a moving cluster the components U'', V'', and W'' are constant and have the same value for all stars belonging to the cluster. We may hence eliminate r from the first equations (2) and get the equation

$$(\gamma_{12}u-\gamma_{11}v)\,U''+(\gamma_{22}u-\gamma_{21}v)\,V''+(\gamma_{32}u-\gamma_{31}v)\,W''=0 \tag{3}$$

or

$$a\,U''+b\,V''+c\,W''=0 \tag{3*}$$

where

$$\begin{aligned} a &= \gamma_{12}u-\gamma_{11}v \\ b &= \gamma_{22}u-\gamma_{21}v \\ c &= \gamma_{32}u-\gamma_{31}v \end{aligned} \tag{4}$$

The value of the coefficients a, b, and c, may be numerically computed for each star, for which the proper motions are known. Hence we obtain an equation of the form (3*) for every star. The quantities U'', V'', and W'' being the same in all equations, we obtain according to the method of least squares the *normal equations*

$$\begin{aligned} (aa)\,U''+(ab)\,V''+(ac)\,W''&=0 \\ (ba)\,U''+(bb)\,V''+(bc)\,W''&=0 \\ (ca)\,U''+(cb)\,V''+(cc)\,W''&=0 \end{aligned} \tag{4*}$$

where

$$(aa)=\sum aa : N, \quad (ab)=\sum ab : N, \text{ etc., where } N=\text{the number of stars.}$$

If D is the determinant of the coefficients in these equations, we get the solution

$$\frac{U''}{D_{aa}}=\frac{V''}{D_{ab}}=\frac{W''}{D_{ac}}=\frac{\sqrt{U''^2+V''^2+W''^2}}{\sqrt{D^2_{aa}+D^2_{ab}+D^2_{ac}}} \tag{4**}$$

where D_{aa}, D_{ab}, D_{ac} denote the subdeterminants of D in respect to the elements (aa), (ab), and (ac).

61. *Geometrical interpretation of* (3).—The equation (3) is the analytical expression of the great circle passing through the proper motion line of a star.

For proving this we consider any point P on the sphere and suppose two great circles PX and PY drawn through P perpendicular to each other.

The tangent of PX at the point P may have the direction cosines

(5) $$\gamma_{11},\ \gamma_{21},\ \gamma_{31}$$

and the tangent of PY in the same point

(5*) $$\gamma_{12},\ \gamma_{22},\ \gamma_{32}$$

so that the normal of the plane containing the great circle PX has the direction cosines (5*) and the normal of PY has the direction cosines (5) (strictly with inverse sign).

The plane containing PX hence has the equation

(6) $$x\gamma_{12}+y\gamma_{22}+z\gamma_{32}=0$$

and that containing PY,

(7) $$x\gamma_{11}+y\gamma_{21}+z\gamma_{31}=0$$

and hence the equation of any great circle through P is

(8) $$\lambda(x\gamma_{12}+y\gamma_{22}+z\gamma_{32})+\mu(x\gamma_{11}+y\gamma_{21}+z\gamma_{31})=0$$

or

$$(\lambda\gamma_{12}+\mu\gamma_{11})x+(\lambda\gamma_{22}+\mu\gamma_{21})y+(\lambda\gamma_{32}+\mu\gamma_{31})z=0$$

where we may suppose that (if the coefficients in this equation coincide with the direction cosines)

$$1=(\lambda\gamma_{12}+\mu\gamma_{11})^2+(\lambda\gamma_{22}+\mu\gamma_{21})^2+(\lambda\gamma_{32}+\mu\gamma_{31})^2=\lambda^2+\mu^2$$

The angle between the normals of (6) and (8) has the cosine

$$\cos\,(68)=\gamma_{12}(\lambda\gamma_{12}+\mu\gamma_{11})+\gamma_{22}(\lambda\gamma_{22}+\mu\gamma_{21})+\gamma_{32}(\lambda\gamma_{32}+\mu\gamma_{31})=\lambda$$

and between (7) and (8)

$$\cos\,(78)=-\gamma_{11}(\lambda\gamma_{12}+\mu\gamma_{11})-\gamma_{21}(\lambda\gamma_{22}+\mu\gamma_{21})-\gamma_{31}(\lambda\gamma_{32}+\mu\gamma_{31})=-\mu.$$

Considering now the proper motions let (6) be the great circle corresponding to the proper motion in $\alpha(u)$, and (7) in $\delta(v)$ and (8) to the total proper motion $\mu=\sqrt{u^2+v^2}$, then we have

$$\cos\,(68)=u:\sqrt{u^2+v^2}=\lambda$$
$$\cos\,(78)=v:\sqrt{u^2+v^2}=-\mu$$

so that the equation of the great circle corresponding to the total proper motion is

$$(\gamma_{12}u-\gamma_{11}v)x+(\gamma_{22}u-\gamma_{21}v)y+(\gamma_{32}u-\gamma_{31}v)z=0$$

62. The solution of the normal equations (4*) corresponds with the determination of the point of intersection of all proper motions of the stars in the cluster.

This point of intersection is called the *convergent* of the cluster. Let A_0 and D_0 be the right ascension and declination of the convergent, and we have

$$\begin{aligned} \alpha_1 = U'' : \Omega &= \cos D_0 \cos A_0 \\ \alpha_2 = V'' : \Omega &= \cos D_0 \sin A_0 \\ \alpha_3 = W'' : \Omega &= \sin D_0 \end{aligned} \tag{9}$$

where

$$\Omega = \sqrt{U''^2 + V''^2 + W''^2} \tag{9*}$$

designates the velocity of the cluster, the sun being supposed at rest.

The values of $U'' : \Omega$, $V'' : \Omega$ and $W'' : \Omega$ are given by (4**).

For determining Ω we may use the third equation (2), which gives, according to (9),

$$W = \Omega(\alpha_1\gamma_{13} + \alpha_2\gamma_{23} + \alpha_3\gamma_{33}) = \Omega \cos \lambda. \tag{10}$$

Here α_1, α_2, α_3 are the direction cosines of Ω and γ_{13}, γ_{23}, γ_{33} the direction cosines of the line of sight W. Hence λ is the angle between the convergent and a star in the cluster.

It is easily found that

$$\begin{aligned} \Omega \sin\lambda &= \sqrt{U^2 + V^2} = r\sqrt{u^2 + v^2} \\ \Omega \cos\lambda &= W \end{aligned}$$

where r is the distance of a star in the cluster. Hence we have

$$r = \frac{W}{\sqrt{u^2 + v^2}}\, tg\, \lambda \tag{11}$$

from which equation the distance of a star in the cluster, and of the cluster itself, may be found.

63. *Numerical application.*—We apply the formulae above to the Hyades, the Taurus cluster, the best known of the moving clusters.

In the memoir of Rasmuson, *L. M.*, 26, the proper motions are given for forty-three stars belonging to the Taurus cluster. Using these values we obtain from the formulae (9) the following values of the direction cosines of the convergent:

$$\begin{aligned} \alpha_1 &= +0.188 \\ \alpha_2 &= +0.961 \\ \alpha_3 &= +0.199 \end{aligned}$$

No strictly accurate computation is made here. The purpose is only to give a rough indication of the method. For more detailed results I refer to the memoir of Rasmuson.

The value of λ, the angle between the convergent and the cluster, may be taken approximately as 30° (for the whole cluster).

We then get for Ω, according to (9*), the value

$$\Omega = 8.65 \text{ Sir/St.}$$

For computing the distance of the cluster we use the following approximate mean values of the proper motions:

$$\bar{u} = +0''.100, \quad \bar{v} = -0''.040$$

which give

$$\sqrt{u^2+v^2} = +0''.108 \text{ per year}$$
$$= +0.524 \text{ radians per stellar year}$$

and now from (11)

$$r = 8.25 \text{ Siriometers}$$

corresponding to a parallax of

$$\pi = \frac{0''.206265}{r} = 0''.025.$$

From the value of the velocity (Ω) and the direction cosines α_1, α_2, α_3 we get

$$U'' = +1.63 \text{ Sir/St.}$$
$$V'' = +8.31 \text{ Sir/St.}$$
$$W'' = +1.71 \text{ Sir/St.}$$

These values give the velocity components in an equatorial system of coördinates. For having the velocity components in a *galactic* system of coördinates, coinciding with that used in the lectures on the velocity of the stars, we have to use the following formulae of reduction

$$U_G = +0.1736\ U'' - 0.9848\ V''$$
$$V_G = +0.4624\ U'' + 0.0815\ V'' + 0.8829\ W''$$
$$W_G = -0.8695\ U'' - 0.1533\ V'' + 0.4695\ W''$$

where U_G, V_G, W_G are the velocity components in the galactic system.

We then get

$$U_G = -7.901 \text{ Sir/St}$$
$$V_G = +2.941 \text{ Sir/St}$$
$$W_G = -1.888 \text{ Sir/St}$$

These are the velocity components of the cluster referred to a galactic system of coördinates having the origin at the sun. For having the "absolute" galactic components we have to add the velocity components of the sun, which according to §11, are

$$U_\odot = +3.37$$
$$V_\odot = +1.45$$
$$W_\odot = +1.62$$

so that the absolute velocity components of the cluster are

$$U''_{abs} = -4.53 = \Omega_{abs} \cos B \cos L$$
$$V''_{abs} = +4.39 = \Omega_{abs} \cos B \sin L$$
$$W''_{abs} = -0.27 = \Omega_{abs} \sin B.$$

From these values we get

$$\begin{aligned} \Omega_{abs} &= + \quad 6.31 \text{ Sir/St} \\ B &= - \quad 1\overset{\circ}{.}8 \\ L &= \quad 135\overset{\circ}{.}9 \end{aligned}$$

We conclude from these numbers that the Taurus cluster is moving nearly parallel to the Galaxy toward a point having a galactic longitude of $135\overset{\circ}{.}9$; a point situated, according to §16, some 25° from the antivertex.

CHAPTER V

DISTRIBUTION OF THE B-STARS

64. The difficulty in the determination of the distribution of the stars in our Galaxy lies principally in our lack of knowledge regarding the general form of the laws of density in the distribution of the stars in space and regarding their absolute brightness or magnitude. For want of such laws the simplest course is to suppose the density to be constant in the whole Galaxy and the absolute brightness to be equal for all stars. This is, indeed, the hypothesis of Herschel, upon which he based his fundamental discussion of the structure of the Milky Way.

The hypothesis is vague and erroneous, especially as regards the density function. This was clear to William Herschel even as he discussed it; but it was possible for him, through such a discussion, to arrive at what may be considered as a first approximation in the solution of the problem.

The next step is to abandon at least one of these assumptions and suppose that either

Hyp. I: D = constant but M varying for different stars; or

Hyp. II: D varying in the Galaxy but M = constant.

Hyposthesis I leads to some interesting conclusions to which we come back later. Of greater importance, and more adequate, is the second hypothesis, especially if the researches are restricted to a special group of stars.

Such a group are the B-stars, for which it is well known from different investigations that the absolute magnitude has a relatively low dispersion and in a first approximation can well be considered as a constant. We will discuss the consequences of such an hypothesis.

65. Before starting our discussion let us first draw some conclusions regarding the absolute magnitude of the stars belonging to the Taurus cluster discussed in the preceding section.

Let

h = apparent luminosity
m = apparent magnitude

of a star.

Then we have between h and m the relation

$$h = C\ 10^{-0.4m} \tag{12}$$

where C is a constant corresponding to the apparent magnitude $m = 0$.

Let h_1 and h_2 be the apparent luminosities of the same star at the distance r_1 and r_2, then

$$\frac{h_1}{h_2} = \frac{r_2^2}{r_1^2} = 10^{-0.4(m_1 - m_2)}$$

Suppose $r_2 = 1$ Siriometer and call the apparent magnitude at this distance the *absolute* magnitude $= M$. Then we get, putting r instead of r_1

$$\frac{1}{r^2} = 10^{-0.4(m-M)}$$

and taking the square root

$$\frac{1}{r} = 10^{-0.2(m-M)} \tag{13}$$

and using the logarithm (Brigg's)

$$m - M = 5 \text{ Log } r, \tag{14}$$

The formula (14) contains the relation between the apparent (m) and the absolute magnitude (M), the distance r being expressed in Siriometers.

Applying this formula to the Taurus cluster we have obtained in §63

$$r = 8.25 \text{ Sir.}$$

so that

$$\text{Log } r = 0.916$$

and from (14) we get

$$M = m - 4.58 \tag{15}$$

supposing all the stars in the cluster at the the same distance.

Let us now take from the table of Rasmuson all stars in the Taurus cluster having the spectral type *Ao*. We then get the table here given, where the first column gives the apparent magnitudes of the stars and the second column their absolute magnitudes computed from (15). The mean absolute magnitude is

m	M
+5.48	+0.90
+5.59	+1.01
+5.27	+0.69
+5.76	+1.18
+4.24	−0.34
+6.39	+1.81
+4.60	+0.02
+5.74	+1.16
+5.97	+1.39
+5.70	+1.12
+5.49	+0.91
+4.85	+0.27
+5.55	+0.97
+5.35	+0.77
+5.43	+0.85
+5.12	+0.54
+5.42	+0.84
Mean	+0.829

$$M = +0.829$$

the deviation from this value being rather small. The disperson, computed from the average deviation, amounts to

$$\sigma = 0.44$$

We may conclude that this value of σ gives an approximate idea of the dispersion of this spectraltype (*Ao*). For the B-stars, especially for the subclasses *Bo*, B_1, etc., there is reason to believe the dispersion is still smaller.

66. To determine the distribution of the B stars we now use the following method. From the previous discussion of the motions of the stars, we know the motion of the sun among the stars. Because of this motion each group of stars in a certain direction gets a definite apparent mean motion equal to

the component of the velocity of the sun in this direction with reverse sign. Comparing this component with the mean proper motions of this group of stars we may derive their absolute magnitude, assuming that all stars in the group have the same absolute brightness. Having determined in this manner the absolute magnitude of the stars it is easy to compute their actual position in space from the coördinates and the apparent magnitude.

67. *Determination of the absolute magnitude from the proper motions.*—Let U and V be the components of the linear velocities of a star in the system K_3 and u and v their proper motions in galactic longitude (multiplied by cos B) and latitude, then we have, according to §37

$$U = ru, \quad V = rv \tag{16}$$

Substituting here from (13) the value

$$r = 10^{0.2(m-M)}$$

and using the abbreviations

$$\begin{aligned} u' &= u\ 10^{0.2\,m} \\ v' &= v\ 10^{0.2\,m} \\ R &= \ 10^{-0.2\,M} \end{aligned} \tag{17}$$

we get, instead of (16),

$$U = R\,u', \quad V = R\,v' \tag{18}$$

The quantities u' and v' are called the *reduced* proper motions of the star. They are equal to the proper motions, if the star is placed at such a distance from the sun that the apparent magnitude of the star is equal to zero. The quantity R is called the reduced distance, being the distance at which the star would have this apparent magnitude (zero).

The values of u' and v' may evidently be computed numerically for each star with known proper motion.

Let us now consider a group of stars all having the same absolute magnitude and hence also the same value of R. Taking the mean value of (18) for all stars in this group in a certain galactic square we get

$$\overline{U} = R\,\overline{u}', \quad \overline{V} = R\,\overline{v}'$$

But the mean values of U and V are equal to the components of the solar velocity with reverse sign for this square. We have called them $U_\odot$ and $V_\odot$ and given their values for different squares in table 6. Hence we have

$$U_\odot = R\,\overline{u}', \quad V_\odot = R\,\overline{v}' \tag{19}$$

Applying these formulae to our 48 squares we get 96 equations from which R may be determined.

It may be noted that u' and v' are the observed quantities and that a certain correlation between R and these quantities cannot be avoided (compare remarks

§41). We may therefore consider in (19) not R but $1/R$ as the unknown quantity and hence the normal equations, according to the method of least squares, take the form

$$(U_o^2) = R(U_o\bar{u}'),\ (V_o^2) = R(V_o\bar{v}') \tag{20}$$

where

$$(U_o^2) = \sum U_o^2 : 48,\ (U_o\bar{u}') = \sum U_o\bar{u}' : 48$$

etc.

More accurately we may combine both the relations (19) and get for R the final equation

$$(U_o^2)+(V_o^2) = R[(U_o\bar{u}')+(V_o\bar{v}')] \tag{21}$$

68. Applying this formula to all B stars brighter than the sixth magnitude and using the proper motions of Boss we get

$$\begin{aligned}(U_o^2) &= +68.402,\quad (V_o^2) = +23.126\\ (U_o\bar{u}') &= +18.076,\quad (V_o\bar{v}') = +\ 5.480\end{aligned}$$

and hence

$$91.528 = R\ 23.556$$

so that

$$R = +3.886$$

which gives

$$M = -5\ \mathrm{Log}\ R = -2.947$$

which is the absolute magnitude of the B stars all being assumed to possess the same absolute brightness.

Using this value of M and the corresponding value of R we can compute the distance of each one of the B stars and determine their distribution in space. Though we would in this manner get an approximate idea of this distribution, a more accurate result may easily be found if suitable consideration is given to the different subtypes of these stars. I have found in *L. M.* II, 14, that these subtypes vary rather considerably in their absolute brightness. The stars of the subtypes B_1 and B_2, characterized by prominent helium lines, are the brightest; those of the subtypes B_3 and B_5 the faintest. In *Bo* bright and faint stars seem to be mixed. The values of R and M obtained for the different subtypes were the following:

Bo :	$R = 3.09$,	$M = -2.45$
B_1 :	8.06,	-4.53
B_2 :	6.83,	-4.17
B_3 :	3.50,	-2.72
B_5 :	3.23,	-2.58

With these values of R the distance r is computed, according to the formula

$$r = R\ 10^{0.2\,m} \tag{22}$$

Applying this formula to each one of the B stars we obtain, regarding their distribution in space, the results given in *L. M.* II, 14. The most interesting of these results are the following.

The B stars form a limited system in the Galaxy with a density gradually diminishing from the center of the system. This center is at a distance from the sun of 18 Siriometers and lies in the direction (equatorial)

$$D = -55°.62, \quad A = 7^h\,42^m$$

or galactic

$$B = -14°.21, \quad L = 237°.90$$

corresponding to the constellation Carina.

The density distribution is the following. Let x, y, z be galactic coördinates. Then the number of B stars at the distance

$$x \pm \frac{1}{2}dx$$

$$y \pm \frac{1}{2}dy$$

$$z \pm \frac{1}{2}dz$$

is

$$D\,dx\,dy\,dz$$

where

$$D = 0.002625\, e^{-\frac{x^2+y^2}{2(37.24)^2} - \frac{z^2}{2(13.09)^2}} \tag{23}$$

The density distribution is characterized by a sphaeroid having the smallest axis directed toward the pole of the Galaxy. The axis in the plane of the Milky Way is, approximately, three times the polar axis.

The number of B stars in the center is

$$0.002625 \text{ per Sir}^3$$

We shall find in the next chapter how these results influence the general problem of the distribution of the stars.

CHAPTER VI

THE PROBLEM OF THE GALAXY

69. *Fundamental equation in stellar statistics.*—Consider some attribute of a star (brightness, color, motion, etc.) and suppose this attribute to have different degrees of intensity for different stars. Let E denote the absolute degree of this intensity and let

$$\varphi(E)\,dE$$

denote the frequency of stars having the degree of intensity

$$E \pm \frac{1}{2} dE$$

We suppose the frequency function defined in such a manner that

$$\int_{-\infty}^{+\infty} dE\ \varphi(E) = 1$$

Observed from the sun (the earth) the attribute may appear to possess the intensity e. We call e the apparent intensity of the attribute. Then generally e is a function of the distance (r) and of E:

$$e = e(E, r); \quad E = E(e, r).$$

We shall deduce an expression of the frequency of stars with an observed intensity within the limits $e \pm \frac{1}{2}\, de$ which I shall call the fundamental equation of stellar statistics.

Consider two concentric spheres of the radii r and $r+dr$. The space between these spheres has the volume

$$4\pi r^2 dr$$

If a portion of this space is cut off by a cone having the apex in the center of the spheres and having a solid angle ω of any form, then the space between the spheres intercepted by the cone has the volume:

$$\omega r^2 dr$$

Let D = density = number of stars per cubic Siriometer. Then the number of stars inside this space is

$$D\omega r^2 dr$$

Of these stars

$$D\omega r^2 dr \times \varphi(E) dE$$

have the absolute intensity E. But

$$dE = \frac{\partial E}{\partial e}\, de$$

because r is here a constant.

Hence the number of stars at the distance r ($\pm \frac{1}{2}\, dr$) with an observed intensity e ($\pm \frac{1}{2}\, de$) is

$$D\omega r^2 dr \varphi(E) \frac{\partial E}{\partial e}\, de$$

Integrating this expression between $r=0$ and $r=\infty$ we get the whole number of stars ($a(e)de$) having an observed intensity e

$$a(e) = \omega \int_0^\infty dr D(r) r^2 \frac{\partial E}{\partial e} \varphi(E)$$

which I call *the fundamental equation of stellar statistics.*

70. *The density function.*—Applying the fundamental equation to the magnitudes of the stars we have to put

$$E = M, \quad e = m$$

and the relation between E and e is

$$M = m - 5 \operatorname{Log} r$$

so that

$$\frac{\partial E}{\partial e} = \frac{\partial M}{\partial m} = 1$$

and we get, if

$$a\,(m)\, dm$$

denotes the number of stars having the apparent magnitude $m \pm \frac{1}{2}\, dm$

$$a(m) = \omega \int_0^\infty dr D(r) r^2 \varphi(m - 5 \operatorname{Log} r). \tag{24}$$

This equation contains three functions

$$a(m), \quad D(r), \quad \varphi(M)$$

and all discussions regarding the distribution of the stars in the Galaxy deal with the determination of these functions, one or all.

It is possible to get $a(m)$ directly. We shall later consider star counts made for its determination. The problem of determining the apparent magnitudes was first solved in the last century, after many difficulties, first visually then photographically. Through the fundamental researches on the brighter stars at Harvard

and Potsdam we now know the magnitudes for most of the stars down to the fifteenth magnitude and fainter. The American observatories (Harvard, Mount Wilson, and others) lead in this regard. We shall later discuss fully the function $a(m)$. I may here point out that from the star counts it is evident that $a(m)$ is a frequency function of the A type which very slightly deviates from the normal form.

Regarding the frequency function $\varphi(M)$ of the absolute magnitudes, the observations are not sufficient at present to make possible a direct determination of its analytical form. We have reason to believe it is nearly of the normal form, but it is very possible that a coefficient of skewness, as well as of excess, has a noteworthy value. Until further investigations clear up the question we may use the normal frequency form even of this function.

It is very difficult to get a suitable form for the density function. It must be observed that, for an analytical, as well as for a numerical, discussion of the distribution of the stars, it is necessary to start from definite assumed forms of the frequency functions for m, M, and D, containing a certain number of parameters which are to be determined from the observations. Regarding the density function a great many suggestions have been proposed which may be summed up in the following six forms:

I $D=\text{constant}; \quad M=\text{constant}$

II $D=\text{constant}; \quad \varphi(M)$ arbitrary

III $D=\gamma r^{-s} \quad (s \text{ positive})$

IV $D=\dfrac{D_o}{\kappa\sqrt{2\pi}}e^{-\frac{(y-y_o)^2}{2\kappa^2}}$, where $y=5 \text{ Log } r$

$(D_o,\ y_o,\ \kappa \text{ constants})$

V $D=\dfrac{C}{r}e^{-k^2r^2}$

VI $D=D_o e^{-\frac{r^2}{2\rho^2}}$

I have worked through all of these six forms of the density function. Regarding these forms the following remarks may be appropriate.

Form I was proposed by Herschel. It can evidently be used only as a rough approximation in connection with a special research on the nearest stars. It leads to a form of $a(m)$ given in §71.

Form II has been used by Schiaparelli and is interesting because it leads to the same form of $a(m)$ as does the hypothesis of Herschel.

Form III was introduced by Seelinger and is the basis of his researches on the distribution of the stars. It leads to an infinite density at the center of the Galaxy (or in the neighborhood of the sun). The corresponding expression of $a(m)$ is given in §73.

Form IV was proposed by Schwarzschild in 1912 and was at the same time used by myself. For $r=0$ we get here $D=0$, which evidently is not consistent with facts; similarly for $D=\infty$ in Form III. The Form IV is, however, of great interest in the theoretical discussion of our problem. We come back to that question in §74.

Form V was used by Dyson and Eddington in 1913. Even here we get, as in Form III, an infinite density at the origin.

Form VI has been used in my researches during the last years on the distribution of the stars. It was suggested to me by the results obtained in 1916 regarding the distribution of the B stars. We get here a finite value of the density at the origin. The form of this density function given above is only a special formulation of the general form discussed in §90.

71. *Hypothesis of Herschel.*—The number of stars—$b(r)$—in a cone of the aperture ω, at the distance r, is, according to §69, given by

$$dr\, b(r) = \omega\, D\, r^2\, dr$$

Hence, for a constant value of D, the number of stars, $N(r)$, nearer than r is given by

$$N(r) = \int_0^r dr\, b(r) = \frac{\omega}{3} D\, r^3$$

so that

$$r = \left(\frac{3N(r)}{\omega D} \right)^{1/3} \tag{25}$$

which is the formula used by Herschel in his star gauges.

Because it is supposed that all stars have the same absolute magnitude (M), we may here substitute, according to (13),

$$r = 10^{\,0.2\,(m-M)}$$

and hence get

$$N(r) = \frac{\omega}{3} D\, 10^{\,0.6\,(m-M)}$$

This is also equal to the number of stars brighter than magnitude m. We designate this number by $A(m)$ and hence have, according to the hypothesis of Herschel

$$A(m) = \frac{\omega}{3} D\, 10^{\,0.6\,(m-M)} \tag{26}$$

From this expression we get

$$A(m) : A(m-1) = 10^{\,0.6} = 3.981 \tag{26*}$$

which would be the constant proportion between the number of stars brighter than two consecutive orders of apparent magnitude.

72. *Hypothesis of Schiaparelli.*—Putting in the equation (24) $D = \text{constant} = D_0$ we get

$$a(m) = D_0 \omega \int_0^\infty dr\, r^2\, \varphi\,(m - 5\ \text{Log}\ r)$$

Substituting here

$$5\ \text{Log}\ r = y = \frac{1}{b} \log r \tag{27}$$

where log r designates the natural logarithm and b is a numerical constant, we get

$$r = e^{by} = 10^{0.2y}$$

so that

$$b = \frac{0.2}{\text{Log}\ e} = 0.460517 \tag{28}$$

Further

$$dr = be^{by}\, dy$$

so that

$$a(m) = \omega\, D_0\, b \int_{-\infty}^{+\infty} dy\, e^{3by}\, \varphi\,(m - y)$$

Putting now

$$-m + y = t; \quad dy = dt$$

we get

$$a(m) = \omega D_0\, be^{3bm} \int_{-\infty}^{+\infty} dt\, e^{3bt}\, \varphi\,(-t)$$

The integral is, however, a constant, independent of m, which form the frequency function $\varphi(-t)$ may have, so that

$$a(m) = \omega C\, b\, e^{3bm}$$

We now obtain

$$A(m) = \int_{-\infty}^{m} dm\, a(m) = \frac{\omega}{3} C\, e^{3bm} = \frac{\omega}{3} C\, 10^{0.6m}$$

so that

$$A(m) : A(m-1) = 3.981$$

as in the hypothesis of Herschel. Hence this law is independent of the frequency distribution of the absolute magnitudes, if only the density is constant.

In order to examine this conclusion we give, from Newcomb-Engelmann, in the following table the number of stars brighter than the six first magnitudes. We find from the third column that the actual proportion is smaller than that drawn from the hypotheses of Herschel and Schiaparelli.

m	$A(m)$	$A(m) : A(m-1)$
1	9	
2	30	3.33
3	75	2.92
4	190	2.67
5	630	3.07
6	1949	3.09

73. *Hypothesis of Seeliger.*—Using the same method as in the preceding section, we get here

$$a(m) = \omega C\, b\, e^{(3-s)bm} = \omega C\, b\, 10^{0.2(3-s)m}$$

and

$$A(m) = \frac{\omega}{3} C\, 10^{0.2(3-s)m}$$

$$A(m) : A(m-1) = 10^{0.2(3-s)} \tag{29}$$

Evidently it is possible here, by a suitable choice of s, to get a better agreement with the observations down to the sixth magnitude. It must, however, be observed that the proportion $A(m):A(m-1)$, even according to this hypothesis, is independent of m and remains the same for all magnitudes; which is not verified by observation.

74. *Hypothesis of Schwarzschild.*—In our fundamental equation

$$a(m) = \omega \int_0^\infty dr\, D\, r^2\, \varphi\,(m - 5 \operatorname{Log} r) \tag{24}$$

three functions are occurring, namely

$$a(m),\ D(r),\ \varphi(M)$$

If any two of these functions are known the third is determined by (24). Such a relation for determining an unknown function is called in mathematics an *integral equation.* The present integral equation has been solved by Schwarzschild (*L. M.* II, 8, §7). The most important consequence of this solution is that the density function for stellar statistics is found to be a normal frequency function of Log r supposing that $a(m)$ is a normal function of m and $\varphi(M)$ a normal function of M.

In §70 we have already shown that we may suitably start the treatment of the stellar problem by assuming these two functions as normal functions. We are hence led to the conclusion that the distribution of the density in the Galaxy ought to follow the law

$$D = \frac{D_0}{\kappa\sqrt{2\pi}}\, e^{-\frac{(y-y_0)^2}{2\kappa^2}} \tag{30}$$

where

$$y = 5 \operatorname{Log} r.$$

If $a(m)$ and $\varphi(M)$ are normal functions this form of D is the only one. We may most easily prove it through substituting (30) in the fundamental equation (24) and assuming for $\varphi(M)$ a normal form, say

$$\varphi(M) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{(M-M_0)^2}{2\sigma^2}} \tag{31}$$

A simple mathematical calculation (it is given in a somewhat abbreviated form in §76) proves that $a(m)$ also takes a normal form, say

$$a(m) = \frac{N}{\alpha\sqrt{2\pi}}\, e^{-\frac{(m-m_0)^2}{2\alpha^2}} \tag{32}$$

where

$$m_0 = M_0 + y_0 + 3b\kappa^2$$
$$\alpha^2 = \sigma^2 + \kappa^2$$

Consequently we ought to use the form (30) of D as soon as the forms (32) and (31) are adopted as the adequate forms of the frequency functions of the apparent and absolute magnitudes.

For this reason I (and other astronomers also), from 1912 on, have worked with this expression of the density function in my researches on the distribution of the stars. It is evident, however, that this frequency distribution cannot be considered as an exact description of the distribution of the stars in space, as a density equal to zero in the neighborhood of the sun is certainly false. After many vain attempts with different density functions I have at last adopted the Form VI which was found valid regarding the distribution of the B stars.

It was, however, necessary to examine the compatibility of such a form with Schwarzschild's theorem on the fundamental equation. The Form VI and a normal distribution of the absolute magnitudes evidently cannot be consistent with a normal distribution of the apparent magnitudes. A direct examination made by Malmqvist in *L. M.* 96 leads to the conclusion that Form VI and a normal distribution of M do not give a normal distribution of m. The corrections to this form are very small being characterized by a skewness of the order $+0.020$ and an excess of the order $+0.007$, and neither correction is in clear conflict with the actual star counts. There is therefore no theoretical objection in using Form VI in a discussion of the distribution of the stars in our Galaxy.

75. Let us hence assume the density of the stars to be distributed according to Law VI and the absolute magnitudes according to the formula (31). To learn the distribution of the stars in the Galaxy it is necessary to determine the following four parameters:

$$D_0,\ \rho,\ M_0,\ \sigma.$$

For this purpose we ought to have four relations between these parameters and known quantities. Three of these relations may be deduced from the fundamental equation

$$a(m) = \omega \int_0^\infty dr\, D\, r^2\, \varphi\,(m - 5\ \mathrm{Log}\ r). \tag{24}$$

We shall deduce the form for these relations. The function $a(m)$ denotes the number of stars having the apparent magnitude m. It is possible, with the help of star counts, to determine:

(1) The total number (N) of stars within the cone having the aperture ω and extending to infinity.

(2) The mean value (m) of the apparent magnitude of these stars.

(3) The dispersion (α) in the distribution of m.

The mathematical expressions of these three quantities—M, m_0, and α—give us the three relations sought.

We have, according to mathematical statistics,

$$\begin{aligned} N &= \int_{-\infty}^{+\infty} dm\, a(m) \\ N m_0 &= \int_{-\infty}^{+\infty} dm\, m\, a(m) \\ N\alpha^2 &= \int_{-\infty}^{+\infty} dm (m - m_0)^2\, a(m) \end{aligned} \tag{33}$$

In these formulae we introduce the value (32) of $a(m)$. Observing now that, according to the definition (31) of $\varphi(M)$, we have

$$\int_{-\infty}^{+\infty} dm\, \varphi\,(m - 5\ \mathrm{Log}\ r) = 1$$

$$\int_{-\infty}^{+\infty} dm\, m\, \varphi\,(m - 5\ \mathrm{Log}\ r) = M_0 + 5\ \mathrm{Log}\ r$$

$$\int_{-\infty}^{+\infty} dm (m - m_0)^2\, \varphi\,(m - 5\ \mathrm{Log}\ r) = \sigma^2 + (M_0 - m_0 + 5\ \mathrm{Log}\ r)^2$$

so that we get from (33)

$$N = \omega \int_0^\infty dr\, D\, r^2 \tag{I}$$

Further

$$Nm_\circ = \omega \int_\circ^\infty dr\, D\, r^2\, (M_\circ + 5\ \mathrm{Log}\ r)$$

$$= N\ M_\circ + \omega \int_\circ^\infty dr\, D\, r^2\, 5\ \mathrm{Log}\ r$$

or

$$\text{(II)} \qquad m_\circ = M_\circ + \int_\circ^\infty dr\, D\, r^2\, 5\ \mathrm{Log}\ r : \int_\circ^\infty dr\, D\, r^2$$

The third equation (33) gives

$$Na^2 = \omega \int_\circ^\infty dr\, D\, r^2\, [\sigma^2 + (M_\circ - m_\circ + 5\ \mathrm{Log}\ r)^2]$$

$$= N\, \sigma^2 + \omega \int_\circ^\infty dr\, D\, r^2\, (M_\circ - m_\circ + 5\ \mathrm{Log}\ r)^2$$

or

$$\text{(III)} \qquad a^2 = \sigma^2 + \int_\circ^\infty dr\, D\, r^2\, (M_\circ - m + 5\ \mathrm{Log}\ r)^2 : \int_\circ^\infty dr\, D\, r^2$$

We may write these three relations in the following form:

$$\text{(34)} \qquad \begin{aligned} &\text{(I)} \quad N = \omega \int_\circ^\infty dr\, D\, r^2 \\ &\text{(II)} \quad m_\circ = M_\circ + m' \\ &\text{(III)} \quad a^2 = \sigma^2 + a'^2 \end{aligned}$$

where

$$m' = \int_\circ^\infty dr\, D\, r^2\, 5\ \mathrm{Log}\ r : \int_\circ^\infty dr\, Dr^2$$

$$a'^2 = \int_\circ^\infty dr\, D\, r^2\, (-m' + 5\ \mathrm{Log}\ r)^2 : \int_\circ^\infty dr\, D\, r^2$$

where evidently a'^2 is a positive quantity.

76. The expressions of N, $m_\circ$, and a^2 of the preceding section are valid for any form of the density function D. Let us first deduce the value of m' and a'^2 if D is supposed to have the form used by Schwarzschild.

$$\text{(30)} \qquad D = C\, e^{-\frac{(y-y_\circ)^2}{2\kappa^2}}$$

where

$$y = 5\ \mathrm{Log}\ r = \frac{1}{b} \log r$$

or

$$r = e^{by},\ dr = be^{by}\, dy$$

Then we get

$$m' = \int_{-\infty}^{+\infty} dy\, y\, e^{3by} D : \int_{-\infty}^{+\infty} dy\, e^{3by} D$$

But we have

$$e^{3by} D = Ce^{+3by_0 + \frac{9}{2}b^2\kappa^2} e^{-\frac{(y-y_0-3b\kappa^2)^2}{2\kappa^2}}$$

Substituting now

$$z = y - y_0 - 3b\kappa^2$$

we get

$$m' = \int_{-\infty}^{+\infty} dz(z+y_0+3b\kappa^2)\, e^{-\frac{z^2}{2\kappa^2}} : \int_{-\infty}^{+\infty} dz\, e^{-\frac{z^2}{2\kappa^2}}$$
$$= y_0 + 3b\kappa^2$$

In like manner we get

$$\alpha'^2 = \int_{-\infty}^{+\infty} dz\, z^2\, e^{-\frac{z^2}{2\kappa^2}} : \int_{-\infty}^{+\infty} dz\, e^{-\frac{z^2}{2\kappa^2}}$$
$$= \kappa^2.$$

so that, for the density function (30), we get for

$$\text{(II)} \quad m_0 = M_0 + y_0 + 3b\kappa^2$$
$$\text{(III)} \quad \alpha^2 = \sigma^2 + \kappa^2$$

as was pointed out in §74.

The Equation (I) here takes the form

$$\text{(I)} \qquad N = b\omega \int_{-\infty}^{+\infty} dy\, e^{3by} D = C\, b\omega\sqrt{2}\pi\kappa\, e^{+3by_0 + \frac{9}{2}b^2\kappa^2}.$$

77. Let us now suppose that the density function has the form VI, so that

$$\text{(35)} \qquad D = D_0 e^{-\frac{r^2}{2\rho^2}}.$$

We then get

$$N = \omega D_0 \int_0^{\infty} dr\, r^2\, e^{-\frac{r^2}{2\rho^2}}$$
$$= \frac{1}{2}\omega D_0 \int_{-\infty}^{+\infty} dr\, r^2\, e^{-\frac{r^2}{2\rho^2}}$$

or

$$\text{(I)} \qquad N = \omega D_0 \sqrt{\frac{\pi}{2}}\rho^3$$

according to the integral formula of Laplace.

Further we have

$$m_0 = M_0 + m'$$

where

$$m' = \int_0^\infty dr\, e^{-\frac{r^2}{2\rho^2}} r^2\, 5 \operatorname{Log} r : \int_0^\infty dr\, e^{-\frac{r^2}{2\rho^2}} r^2$$

or introducing

$$r = \rho\, s; \quad dr = \rho ds$$

$$m' = \int_0^\infty ds\, s^2 e^{-\frac{s^2}{2}} (5 \operatorname{Log} \rho + 5 \operatorname{Log} s) : \int_0^\infty ds\, s^2\, e^{-\frac{s^2}{2}}$$

or

$$m' = 5 \operatorname{Log} \rho + A$$

where

$$A = \int_0^\infty ds\, s^2 e^{-\frac{s^2}{2}}\, 5 \operatorname{Log} s : \int_0^\infty ds\, e^{-\frac{s^2}{2}} s^2 \tag{36}$$

which evidently is a numerical constant.

Our third equation is

$$\alpha^2 = \sigma^2 + \alpha'^2$$

Introducing in α'^2 instead of r the value ρs, we get

$$\begin{aligned} \alpha'^2 &= \int_0^\infty ds\, e^{-\frac{s^2}{2}} s^2 (-A + 5 \operatorname{Log} s)^2 : \int_0^\infty ds\, s^2 e^{-\frac{s^2}{2}} \\ &= A^2 - 2A \int_0^\infty ds\, e^{-\frac{s^2}{2}} 5 \operatorname{Log} s : \int_0^\infty ds\, s^2 e^{-\frac{s^2}{2}} + B \\ &= -A^2 + B \end{aligned}$$

where

$$B = \int_0^\infty ds\, s^2 e^{-\frac{s^2}{2}} (5 \operatorname{Log} s)^2 : \int_0^\infty ds\, s^2 e^{-\frac{s^2}{2}} \tag{37}$$

The values of the integrals A and B have been computed by Amanuensis Ohlsson who has obtained

$$A = +0.792193$$
$$B = +1.729537$$

so that

$$\alpha'^2 = B - A^2 = +1.101967.$$

Our three formulae for determining D_0, ρ, M_0, and σ hence have the following appearance:

$$\begin{aligned} &\text{(I)} \quad N = \omega \sqrt{\frac{\pi}{2}} D_0\, \rho^3 \\ &\text{(II)} \quad m_0 = M_0 + 5 \operatorname{Log} \rho + 0.792193 \\ &\text{(III)} \quad \alpha^2 = \sigma^2 + 1.101967 \end{aligned} \tag{38}$$

CHAPTER VII

FREQUENCY DISTRIBUTION OF THE APPARENT MAGNITUDES OF THE STARS

A. Determinant Method

78. *Abridged method for computing the characteristics of a frequency function.*— Let $\varphi(x)$ be the normal function, so that

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

and let $Q(x)$ be the probability integral defined by

$$Q(x) = \int_{-\infty}^{x} dx\, \varphi(x)$$

then we know that $Q(-\infty)=0$, $Q(0)=\frac{1}{2}$, $Q(+\infty)=+1$.

Putting

(39) $$y = Q(x)$$

we may denote the inverse function of $Q(x)$ by $Q^{-1}(y)$, so that

(39*) $$x = Q^{-1}(y),$$

then we have

$$Q^{-1}(0) = -\infty, \quad Q^{-1}(\tfrac{1}{2}) = 0, \quad Q^{-1}(+1) = +\infty.$$

Evidently,

(40) $$Q(-x) = 1 - Q(x)$$

and

(40*) $$Q^{-1}(1-y) = -Q^{-1}(y)$$

Suppose now that we consider some attribute of an individual and let x be the intensity of this attribute. Let $A(x)$ be the number of individuals having this attribute with an intensity smaller than x. And let N, m_o, and a be the characteristics of the frequency function of x. Then we have

$$A(x)=\frac{N}{a\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{(x-m_0)^2}{2a^2}}dx$$

$$=\frac{N}{\sqrt{2\pi}}\int_{-\infty}^{\frac{x-m_0}{a}} dt\, e^{-\frac{t^2}{2}}$$

or

$$A(x)=NQ\left(\frac{x-m_0}{a}\right) \tag{41}$$

and

$$\frac{x-m_0}{a}=Q^{-1}\left(\frac{A(x)}{N}\right) \tag{41*}$$

According to this equation, it is mathematically possible to determine the three characteristics N, m_0, and a if we know $A(x)$ for *three* different values of x, say x_1, x_2, and x_3. It is this method we shall examine here, applying it to the frequency function of the apparent magnitudes.

Let us first consider a simple problem. Suppose the number N of individuals belonging to a given population to be known. Then we need to know $A(x)$ only for two values of x, say x_1 and x_2, and we get the equations

$$\frac{x_1-m_0}{a}=Q^{-1}\left(\frac{A(x_1)}{N}\right)$$

$$\frac{x_2-m_0}{a}=Q^{-1}\left(\frac{A(x_2)}{N}\right)$$

Subtracting we get

$$a=\frac{x_1-x_2}{Q^{-1}\left(\frac{A(x_1)}{N}\right)-Q^{-1}\left(\frac{A(x_2)}{N}\right)} \tag{42}$$

and then we have

$$m_0=x_1-aQ^{-1}\left(\frac{A(x_1)}{N}\right)=x_2-aQ^{-1}\left(\frac{A(x_2)}{N}\right) \tag{42*}$$

The characteristics a and m_0 are given here by two simple linear equations.

The function $Q(x)$ is tabulated in my *Grundzüge der mathematischen Statistik*. The same table may be used for obtaining the values of $Q^{-1}(y)$.

Numerical example.—In some measures made by C. M. Fürst and F. C. C. Hansen of the index of skulls from Greenland (*Grundzüge*, p. 103) it was found that of these skulls 84 per cent had an index smaller than 75 (Dolichocephalic) and 15 per cent an index between 75 and 80 (Mesocephalic). Putting $x_1=75$, $x_2=80$ we have in this case

$$\frac{A(x_1)}{N}=0.84;\quad \frac{A(x_2)}{N}=0.90$$

We then get from (42) and (42*)

$$\sigma = 3.7, \quad M = 71.3 \text{ (dolichocephalic)}.$$

In this case we do not use the total number of individuals. It is sufficient to know only a percentage of the material.

79. *Apparent star magnitudes.*—In the star counts we do not know the total number (N) of stars (per square degree). That is to be determined from our counts of the number of stars of different apparent magnitudes.

The number—$A(m)$—of stars brighter than magnitude m is given by the formula

$$A(m) = \int_{-\infty}^{m} dm\, a(m)$$

$$= N \int_{-\infty}^{m} dm \frac{e^{-\frac{(m-m_o)^2}{2a^2}}}{a\sqrt{2\pi}}$$

or

$$A(m) = NQ\left(\frac{m-m_o}{a}\right) \tag{43}$$

so that

$$\frac{m-m_o}{a} = Q^{-1}\left(\frac{A(m)}{N}\right) \tag{43*}$$

Supposing now $A(m)$ to be known for three values of m, say m_1, m_2, and m_3, and put

$$A_1 = A(m_1), \quad A_2 = A(m_2), \quad A_3 = A(m_3)$$

then we get

$$\frac{m_1-m_o}{a} = Q^{-1}(A_1/N)$$

$$\frac{m_2-m_o}{a} = Q^{-1}(A_2/N)$$

$$\frac{m_3-m_o}{a} = Q^{-1}(A_3/N)$$

Putting

$$x = m_o/a, \quad y = 1/a \tag{44}$$

we get the following equations for determining x, y, and N

$$\begin{aligned} -x + m_1 y &= Q^{-1}(A_1/N) \\ -x + m_2 y &= Q^{-1}(A_2/N) \\ -x + m_3 y &= Q^{-1}(A_3/N) \end{aligned} \tag{45}$$

Eliminating x and y we get the following equation, containing only N as unknown quantity:

$$(46) \qquad f(N) = \begin{vmatrix} -1, & m_1, & Q^{-1}(A_1/N) \\ -1, & m_2, & Q^{-1}(A_2/N) \\ -1, & m_3, & Q^{-1}(A_3/N) \end{vmatrix} = 0$$

This equation, which it is impossible to solve analytically, must be solved through trials giving a series of values to N, till the determinant changes its sign.

Developing the determinant we get

$$(46^*) \qquad f(N) = (m_2 - m_3)Q^{-1}(A_1/N) + (m_3 - m_1)Q^{-1}(A_2/N) + (m_1 - m_2)Q^{-1}(A_3/N) = 0$$

80. *Star counts in Lund.*—For computing N according to this equation it is necessary to know the number $A(m)$ of stars down to stars of a sufficiently faint apparent magnitude. In my first attempt (1911) in this direction I based the computation on star counts in Argelander's *Bonner Durchmusterung* combined with results obtained in Paris from their work on the "Carte du ciel." The resulting values of N are given below in §83.

Leaving out of consideration the fact here that results could be obtained only from a limited part of the sky, it was soon found that these attempts failed because of the impossibility of getting available values of the faint star magnitudes. Since that time, however, this difficulty of this question has been lessened, chiefly through the extensive researches at Harvard and Mount Wilson.

Even the extension of star maps has considerably developed during the last fifteen years. The Harvard map gives the stars of the whole sky down to the eleventh magnitude and the Franklin-Adams charts extend to the fourteenth magnitude and fainter. Further, the energetic measurements of the Kapteyn areas give, together with the embracing observations of the Harvard observatory, sufficient material for the reduction and correction of photographic star counts in any part of the sky.

The star count of the Harvard map was made by Henie at the observatory of Lund in the year 1913, of the Franklin-Adams charts by the staff of the same observatory in 1922. Is is therefore now possible to determine fairly well the characteristics of the frequency distribution of the apparent magnitudes of the stars.

81. *Stars of the sixth magnitude.*—The stars of the sixth magnitude and brighter have all been photometrically determined by Pickering so far as the luminosity is measured by the eye. In diagram 6 I give for each one of the forty-eight galactic squares the total number of stars visually brighter than the sixth magnitude and, in small characters, the corresponding number per square degree (1 square = 859.436 square degrees). The total number of stars visually brighter than the sixth magnitude is 4707, corresponding to 0.1141 stars per square degree.

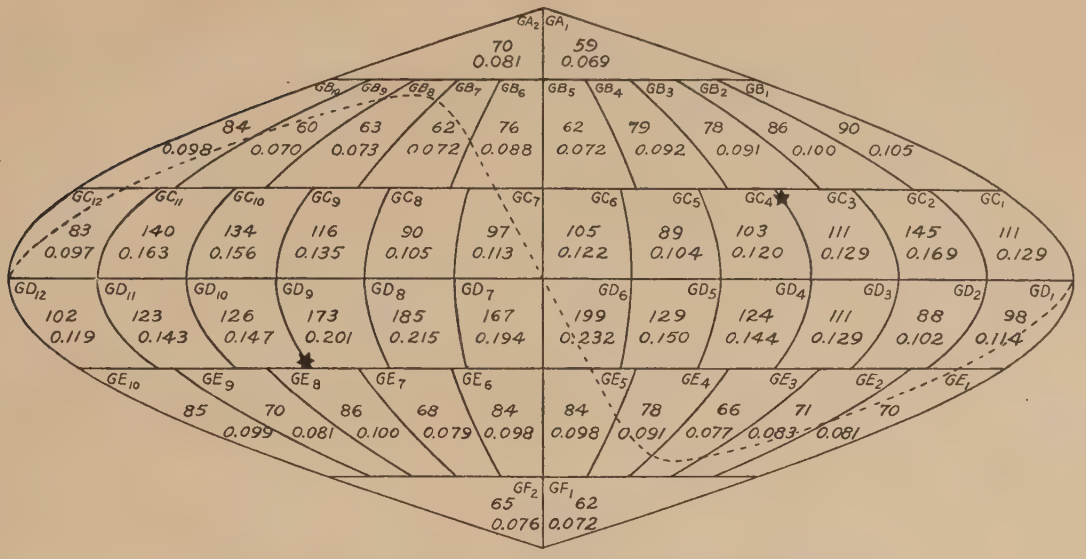


Diagram 6.—Number of Stars Visually Brighter than the Sixth Magnitude.

Regarding the fainter stars the visual magnitude is known only for some individual stars (at least for $m'>9$), the observations being made photographically so that the photographic magnitude is easier to determine. The photographic magnitude of the brighter stars is, however, very incompletely known. It may be at least roughly determined from the visual magnitudes with the help of the color index. In diagram 7 I give the number of stars in the different galactic squares photographically brighter than the sixth magnitude, the color index being taken from my memoir, *L. M.* II, 19.

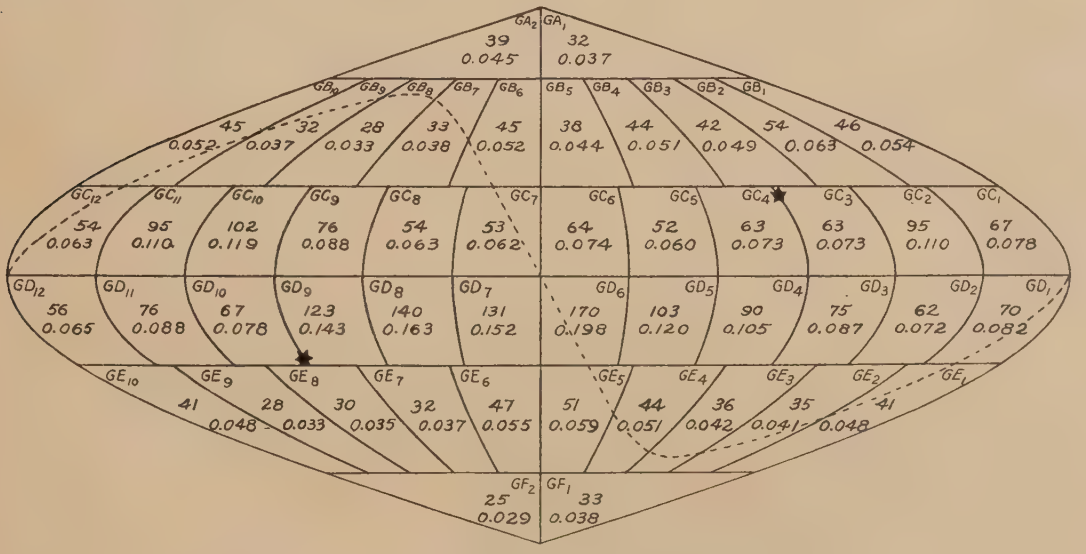


Diagram 7.—Number of Stars Photographically Brighter than the Sixth Magnitude.

Here also the numbers per square degree are given in small characters. The total number of stars photographically brighter than the sixth magnitude amounts to 2922 corresponding to 0.0708 per square degree.

In the following table I summarize the results from these diagrams.

TABLE 13

Number of stars brighter than the sixth magnitude

	visually		photographically	
	total	per □°	total	per □°
A-squares	129	0.075	71	0.041
B-squares	740	0.086	407	0.047
C-squares	1324	0.128	838	0.081
D-squares	1625	0.158	1163	0.113
E-squares	762	0.089	385	0.045
F-squares	127	0.074	58	0.034
Whole sky	4707	0.1141	2922	0.0708

The diagrams as well as the table show that the number of stars brighter than the sixth magnitude gradually increases from the Poles toward the Milky Way. Especially in regard to the visual magnitudes, the number in the Milky Way is nearly double the number at the Poles.

It is further remarkable that the greatest number of stars per square degree occurs in the D squares D_6 to D_9, a fact which probably may be explained by the location of the center of the Galaxy in this direction as already found from the distribution of the B stars.

82. *Distribution of the apparent magnitudes at the North Galactic Pole.*—It is not possible to give here a survey of the star counts made at Lund because these counts are at present not finished. As an illustration of the method explained in the preceding paragraphs I give, however, preliminary results regarding the star gauges around the galactic North Pole.

We have the following values for the number of stars $A(m)$ per square degree brighter than the sixth, twelfth, and sixteenth photographic magnitudes. The photographic magnitude is denoted by m'.

$$m'_1=6; \quad A(m'_1)=0.041$$
$$m'_2=12; \quad A(m'_2)=15.0$$
$$m'_3=16; \quad A(m'_3)=180$$

We have now to compute the value of the determinant $f(N)$ according to the formula (46*) giving to N a series of values till $f(N)$ vanishes. The value of $Q^{-1}(y)$ is obtained from the specially prepared table 14.

TABLE 14

Table for $Q^{-1}(y)$

log. y	$Q^{-1}(y)$	Δ	log. y	$Q^{-1}(y)$	Δ
0.00−10	−6.361		2.60−10	−5.368	
		35			42
0.10−10	−6.326		2.70−10	−5.326	
		36			42
0.20−10	−6.290		2.80−10	−5.284	
		36			42
0.30−10	−6.254		2.90−10	−5.242	
		36			43
0.40−10	−6.218		3.00−10	−5.199	
		36			43
0.50−10	−6.182		3.10−10	−5.156	
		36			43
0.60−10	−6.146		3.20−10	−5.113	
		37			44
0.70−10	−6.109		3.30−10	−5.069	
		37			44
0.80−10	−6.072		3.40−10	−5.025	
		37			44
0.90−10	−6.035		3.50−10	−4.981	
		37			45
1.00−10	−5.998		3.60−10	−4.936	
		38			45
1.10−10	−5.960		3.70−10	−4.891	
		38			45
1.20−10	−5.923		3.80−10	−4.846	
		38			46
1.30−10	−5.885		3.90−10	−4.800	
		39			47
1.40−10	−5.846		4.00−10	−4.753	
		38			46
1.50−10	−5.808		4.10−10	−4.707	
		39			47
1.60−10	−5.769		4.20−10	−4.660	
		39			48
1.70−10	−5.730		4.30−10	−4.612	
		39			48
1.80−10	−5.691		4.40−10	−4.564	
		39			49
1.90−10	−5.652		4.50−10	−4.515	
		40			49
2.00−10	−5.612		4.60−10	−4.466	
		40			50
2.10−10	−5.572		4.70−10	−4.416	
		40			50
2.20−10	−5.532		4.80−10	−4.366	
		41			51
2.30−10	−5.491		4.90−10	−4.315	
		41			51
2.40−10	−5.450		5.00−10	−4.264	
		41			50
2.50−10	−5.409		5.10−10	−4.214	
		41			52

TABLE 14—Continued

Table for $Q^{-1}(y)$

log. y	$Q^{-1}(y)$	△	log. y	$Q^{-1}(y)$	△
5.20−10	−4.162		7.60−10	−2.654	
		52			79
5.30−10	−4.110		7.70−10	−2.575	
		54			81
5.40−10	−4.056		7.80−10	−2.494	
		55			82
5.50−10	−4.001		7.90−10	−2.412	
		55			86
5.60−10	−3.946		8.00−10	−2.326	
		56			87
5.70−10	−3.890		8.10−10	−2.239	
		56			91
5.80−10	−3.834		8.20−10	−2.148	
		57			94
5.90−10	−3.777		8.30−10	−2.054	
		58			96
6.00−10	−3.719		8.40−10	−1.958	
		59			101
6.10−10	−3.660		8.50−10	−1.857	
		59			104
6.20−10	−3.601		8.60−10	−1.753	
		60			109
6.30−10	−3.541		8.70−10	−1.644	
		61			115
6.40−10	−3.480		8.80−10	−1.529	
		63			120
6.50−10	−3.417		8.90−10	−1.409	
		63			127
6.60−10	−3.354		9.00−10	−1.282	
		64			136
6.70−10	−3.290		9.10−10	−1.146	
		65			145
6.80−10	−3.225		9.20−10	−1.001	
		67			157
6.90−10	−3.158		9.30−10	−0.844	
		68			173
7.00−10	−3.090		9.40−10	−0.671	
		69			193
7.10−10	−3.021		9.50−10	−0.478	
		70			220
7.20−10	−2.951		9.60−10	−0.258	
		72			261
7.30−10	−2.879		9.70−10	+0.003	
		74			331
7.40−10	−2.805		9.80−10	+0.334	
		74			488
7.50−10	−2.731		9.90−10	+0.822	
		77			
			10.00−10	$+\infty$	

We then get the following numerical results supposing $N=500$, 1000, 1500, 1650.

N	500	1000	1500	1650
Q_1^{-1}	-3.767	-3.939	-4.036	-4.059
Q_2^{-1}	-1.880	-2.170	-2.326	-2.362
Q_3^{-1}	-0.358	-0.915	-1.175	-1.232
$f(N)$	-1.57	-0.45	-0.07	$+0.01$

It follows from these numbers that the function $f(N)$ vanishes for a value of N between $N=1500$ and $N=1650$. (If the material were more complete it would be necessary to interpolate for N.) Suppose the solution to be

$$N=1650$$

We now get the following formulae for determining x and y

$$\begin{aligned} -x+\ 6\,y&=-4.059\\ -x+12\,y&=-2.362\\ -x+16\,y&=-1.232 \end{aligned}$$

which give

$$x=+5.755,\quad y=+0.2827$$

According to (44) we now get

$$m_0=+20.36;\quad a=+3.54$$

The result, expressed in words, is that at the north galactic pole the total number of stars in the Galaxy amounts to 1650 stars per square degree, the mean photographic magnitude of these stars is +20.36 with a dispersion of 3.54 magnitudes. We conclude that about 1/6 of the stars are brighter than

$$m'=20.36-3.54=16.82.$$

With a photographic exposure giving stars down to the twentieth magnitude we would obtain nearly half of all the stars in this direction.

With the formula

$$A(m')=NQ\left(\frac{m'-m_0}{a}\right)$$

we may compute the number of stars down to each arbitrary magnitude. The result down the the twentieth magnitude is given in the table below.

TABLE 15

Number of stars at the north galactic pole down to different photographic magnitudes

$N=1650$, $m_o=20.36$, $\sigma=3.537$

m'	$A(m')$	Observed	$A(m'):A(m'-1)$
5	0.012	0.016	.....
6	0.041	0.041	3.4
7	0.13	0.15	3.2
8	0.40	0.43	3.0
9	1.10	1.17	2.8
10	2.80	2.6	2.5
11	6.64	6	2.3
12	15.1	15	2.2
13	31.0	35	2.0
14	58.3	62	1.9
15	106.0	110	1.8
16	180.4	180	1.7
17	282	.....	.....
18	415	.....	.....
19	581	.....	.....
20	759	.....	.....

The agreement between the calculated and observed numbers is good. For $m'=6$, 12, and 16 it is complete as it should be. For other magnitudes the difference may be well explained by accidental deviations from the normal form of frequency. The proportion between the number of stars of two consecutive magnitudes diminishes from 3.4 at the sixth magnitude to 1.3 at the twentieth magnitude.

83. In the year 1911 I computed by this method N, m_o, and a for 9 squares (equatorial squares) and got the following results:

Square	N per □°	m_o	a
C_1	2310	18.3	3.02
C_2	3971	19.9	3.36
C_3	5684	18.9	3.18
C_4	35000	20.1	3.12
C_5	1950	17.5	2.83
C_6	698	16.9	2.85
C_7	733	17.6	3.11
C_8	1292	18.2	3.18
C_9	1804	16.9	2.70

These results were based on star counts from *B. D.* for $m=6$ and 9.2 and from *C. d. C.* for $m=13.89$ (visual). The last limiting magnitude was obtained from nine magnitude observations at Harvard valid for comparison stars in Pater Hagen's charts for variable stars.

All three of these limiting magnitudes are rather uncertain.

Application of the same method to other parts of the sky very often gave $N=\infty$ and in other respects the method was found to be very uncertain. During the past ten years I have made one attempt after another and performed many hundreds (perhaps thousands) of such computations. The possibility of obtaining a definite answer seemed very small until more recently I developed finally a method that leads to acceptable results. I also now use better star gauges as the basis of the calculations. This method is developed in the next chapter.

B. Tangent Method

84. *Development of the Q function.*—We have

$$Q(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} dt\, e^{-\frac{t^2}{2}} \tag{47}$$

$$=\frac{1}{2}+\frac{1}{\sqrt{2\pi}}\int_{0}^{x} dt\, e^{-\frac{t^2}{2}}$$

Supposing x small we may develop $Q(x)$ into powers of x. We have

$$e^{-\frac{t^2}{2}}=1-\frac{t^2}{2}+\frac{1}{\underline{|2}}\frac{t^4}{2^2}-\frac{1}{\underline{|3}}\frac{t^6}{2^3}+\frac{1}{\underline{|4}}\frac{t^8}{2^4}-\cdots.$$

so that

$$Q(x)=\frac{1}{2}+\frac{1}{\sqrt{2\pi}}\left\{x-\frac{x^3}{6}+\frac{x^5}{40}-\frac{x^7}{336}+\frac{x^9}{3456}-\cdots.\right\}$$

which expansion is formally valid for all values of x but numerically can be used only if x is rather small.

For great values of x a suitable expansion of $Q(x)$ is given by Bruns (*Wahrscheinlichkeitsrechnung und Kollektivmasslehre*, p. 40). Put

$$F_n(x)=\int_{x}^{\infty} dt\, \frac{e^{-\frac{t^2}{2}}}{\sqrt{2\pi}}\, t^{-2n} : \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} \tag{48}$$

$$=\int_{x}^{\infty} dt\, \varphi(t)\, t^{-2n} : \varphi(x)$$

which gives

$$F_0(x) = \int_x^\infty dt\ \varphi(t) \ : \ \varphi(x)$$
$$= (1 - Q(x)\ ) \ : \ \varphi(x)$$

so that

(49) $$Q(x) = 1 - \varphi(x)\ F_0(x)$$

To obtain the form of $F_0(x)$ for great values of x we observe that

$$\frac{d(e^{-\frac{t^2}{2}} t^{-(2n+1)})}{dt} = -e^{-\frac{t^2}{2}}\ t^{-2n} - (2n+1)\ e^{-\frac{t^2}{2}}\ t^{-(2n+2)}$$

or multiplying by $1 : \sqrt{2\pi}$ and integrating between x and $+\infty$

$$-\varphi(x)\ x^{-(2n+1)} = -\int_x^\infty dt\ \varphi(t)\ t^{-2n} - (2n+1) \int_x^\infty \varphi(t)\ t^{-(2n+2)}$$

or

$$x^{-(2n+1)} = F_n(x) + (2n+1)\ F_{n+1}(x)$$

or

(50) $$F_n(x) = x^{-(2n+1)} - (2n+1)\ F_{n+1}(x)$$

We here put $n = 0, 1, 2, \ldots\ldots$ and we get

$$F_0(x) = x^{-1} - F_1(x)$$
$$F_1(x) = x^{-3} - 3F_2(x)$$
$$F_2(x) = x^{-5} - 5F_3(x)$$
$$F_{n-1}(x) = x^{-(2n-1)} - (2n-1)\ F_n(x)$$
$$F_n(x) = x^{-(2n+1)} - (2n+1)\ F_{n+1}(x)$$

so that

$$F_0(x) = \frac{1}{x} - F_1(x)$$
$$-F_1(x) = -\frac{1}{x^3} + 1.3\ F_2(x)$$
$$+1.3\ F_2(x) = \frac{3}{x^5} - 3.5\ F_3(x)$$
$$-3.5\ F_3(x) = -\frac{3.5}{x^7} + 3.5.7.F_4(x)$$
$$\mp 3.5 \ldots (2n-3)\ F_{n-1}(x) = \mp \frac{3.5 \ldots 2n-3}{x^{2n-1}} \pm 3.5 \ldots \overline{2n-1}\ F_n(x)$$
$$\pm 3.5 \ldots (2n-1)\ F_n(x) = \pm \frac{3.5 \ldots \overline{2n-1}}{x^{2n+1}} \mp 3.5 \ldots \overline{2n+1}\ F_{n+1}(x)$$

and adding

$$F_0(x) = \frac{1}{x} - \frac{1}{x^3} + \frac{3}{x^5} - \frac{3.5}{x^7} + \ldots \pm \frac{3.5 \ldots \overline{2n-1}}{x^{2n+1}} \mp \text{Remainder}$$

so that, according to (49),

(51) $$Q(x) = 1 - \varphi(x) \left\{ \frac{1}{x} - \frac{1}{x^3} + \frac{3}{x^5} - \frac{3.5}{x^7} + \ldots \mp \frac{3.5 \ldots \overline{2n-1}}{x^{2n+1}} \right\} + \text{Residuum},$$

where

$$\text{Residuum} = \pm \varphi(x)\ 3.5 \ldots . \overline{2n+1}\ F_{n+1}(x)$$

For evaluating the amount of this residuum, corresponding to the error in the value of $Q(x)$ when the series is cut off at the term multiplied by $x^{-(2n+1)}$, we observe, with Bruns, that

$$F_{n+1}(x) = \int_x^\infty dt\ \varphi(t)\ t^{-(2n+2)} : \varphi(x)$$

Substituting in the integral for $\varphi(t)$ its maximum value, $\varphi(x)$, we get

$$F_{n+1}(x) = \theta \int_x^\infty \frac{dt}{t^{2n+2}} = \frac{\theta}{(2n+1)\ t^{2n+1}} \tag{52}$$

where $|\theta| < 1$.

In words: The error in computing the value of $Q(x)$ from the formula (51) is always smaller than the last term of the series taken into consideration.

We may especially put

$$Q(x) = 1 - \varphi(x)\left\{\frac{1}{x} - \frac{1}{x^3}\right\} \tag{51*}$$

or more exactly

$$Q(x) = 1 - \varphi(x)\left\{\frac{1}{x} - \frac{1}{x^3} + \frac{3}{x^5}\right\} \tag{51*}$$

We may consider (51) as a development of $Q(x)$ in the neighborhood $x = +\infty$. For $x = +\infty$ we get, from (51)

$$Q(+\infty) = +1$$

in accordance with the integral expression.

If, however, we put $x = -\infty$, we get from (51)

$$Q(-\infty) = +1$$

whereas, according to the integral formula (47), we ought to have

$$Q(-\infty) = 0$$

The formula (51) is hence valid only for great *positive* values of x.

To get a formula valid for high *negative* values of x we observe that, according to (40),

$$Q(-x) = 1 - Q(x)$$

or

$$Q(-x) = \varphi(x)\left\{\frac{1}{x} - \frac{1}{x^3} + \frac{3}{x^5} - \frac{15}{x^7} + \ldots\right\}$$

so that for the negative value of x

$$Q(x) = -\varphi(x)\left\{\frac{1}{x} - \frac{1}{x^3} + \frac{3}{x^5} - \frac{15}{x^7} + \ldots\right\} \tag{53}$$

Summing up we have the following expressions of $Q(x)$:

For small (positive or negative) values of x:

$$(A)\quad Q(x)=\frac{1}{2}+\frac{1}{\sqrt{2\pi}}\left\{x-\frac{x^3}{6}+\frac{x^5}{40}-\frac{x^7}{336}+\frac{x^9}{3456}-\cdots\right\}$$

For great positive values of x:

$$(B)\quad Q(x)=1-\varphi(x)\left\{\frac{1}{x}-\frac{1}{x^3}+\frac{3}{x^5}-\frac{15}{x^7}+\cdots\right\}$$

For great negative values of x:

$$(C)\quad Q(x)=-\varphi(x)\left\{\frac{1}{x}-\frac{1}{x^3}+\frac{3}{x^5}-\frac{15}{x^7}+\cdots\right\}$$

The errors in (B) or (C) are smaller than the last term taken into consideration.

85. Instead of using the "determinant-method" of the preceding chapter to determine the characteristics of the frequency function, we may take into consideration for this purpose the form of this function. We therefore proceed in the following manner:

Put

(54) $$y=\text{Log } Q(x)=0.4343 \log Q(x)$$

where $0.4343=\text{Modulus}=\text{Log } e$.

We then have

$$\frac{dy}{dx}=0.4343\frac{\varphi(x)}{Q(x)}$$

so that for great negative values (the only ones we have to consider here) we have, according to formula (C) of the preceding section,

(55) $$\frac{dy}{dx}=-\frac{0.4343}{\frac{1}{x}-\frac{1}{x^3}+\frac{3}{x^5}-\frac{15}{x^7}+\cdots}$$

$$=-\frac{0.4343\,x}{1-\frac{1}{x^2}+\frac{3}{x^4}-\frac{15}{x^6}+\cdots}$$

By means of this formula we are able to compute the angle that the curve (54) makes with the X-axis for any value of x.

86. Applying this result to the frequency of the apparent magnitudes we observe that, according to (41), the number of stars brighter than the apparent magnitude m is given by the formula

(41) $$A(m)=NQ\left(\frac{m-m_0}{a}\right)=NQ(X)$$

$$X=\frac{m-m_0}{a}$$

so that

$$\text{Log } A(m)=\text{Log } N+\text{Log } Q(X)$$

and

$$\frac{d\text{ Log } A(m)}{dm}=\frac{1}{a}\quad\frac{d\text{ Log } Q(X)}{dX}\ .$$

According to (55),

$$\frac{d \operatorname{Log} A(m)}{dm} = -\frac{1}{\alpha} \quad \frac{0.4343\, X}{1 - \frac{1}{X^2} + \frac{3}{X^4} - \frac{15}{X^6} + \ldots}$$

or writing

(56) $$p = \frac{1}{0.4343} \quad \frac{d \operatorname{Log} A(m)}{dm} = 2.3026 \quad \frac{d \operatorname{Log} A(m)}{dm}$$

we get

(57) $$\alpha p = -\frac{X}{1 - \frac{1}{X^2} + \frac{3}{X^4} - \frac{15}{X^6} + \ldots}$$

which formula may be inverted and gives

(57*) $$X = -\alpha p \left\{ 1 - \frac{1}{\alpha^2 p^2} + \frac{1}{\alpha^4 p^4} - \ldots . \right\}$$

Introducing here the value of X

$$X = \frac{m - m_0}{\alpha}$$

we get

(58) $$m - m_0 = -\alpha^2 p \left\{ 1 - \frac{1}{\alpha^2 p^2} + \frac{1}{\alpha^4 p^4} - \ldots . \right\} .$$

Let us now suppose that for two values, m_1 and m_2, the value of p has been determined, say p_1 and p_2, then we get, if terms of the order $1/p^3$ are neglected,

(59) $$\begin{cases} m_0 - m_1 = \alpha^2 p_1 - \dfrac{1}{p_1} \\ m_0 - m_2 = \alpha^2 p_2 - \dfrac{1}{p_2} \end{cases}$$

and

$$m_2 - m_1 = \alpha^2 (p_1 - p_2) - \frac{1}{p_1} + \frac{1}{p_2}$$

which gives

(60) $$\alpha^2 = \frac{m_2 - m_1}{p_1 - p_2} - \frac{1}{p_1 p_2} .$$

By this formula the dispersion α of the apparent magnitude may be determined as soon as p_1 and p_2 are known.

As soon as α is found we get the value of m_0 from (59).

Finally to determine N we compute the value of

$$Q\left(\frac{m - m_0}{\alpha}\right)$$

and get according to (41)

(61) $$N = A(m) : Q\left(\frac{m - m_0}{\alpha}\right)$$

The problem is then solved.

87. We have supposed in the preceding section the value of p is known for two values of m. We shall now show how this determination may be arrived at.

We have

$$p = 2.3026 \frac{d \operatorname{Log} A(m)}{dm}$$

Instead of differentials we may, if the interval is small, use differences and put

$$\begin{aligned} \frac{d \operatorname{Log} A(m)}{dm} &= \frac{\triangle \operatorname{Log} A(m)}{\triangle m} \\ &= \frac{\operatorname{Log} A(m+\triangle m) - \operatorname{Log} A(m)}{\triangle m} \\ &= \frac{1}{\triangle m} \operatorname{Log} \frac{A(m+\triangle m)}{A(m)} \end{aligned}$$

If the variation in $A(m)$ with m is rather slow we may put $\triangle m = 1$ and get

$$\frac{d \operatorname{Log} A(m)}{dm} = \operatorname{Log} \frac{A(m+1)}{A(m)}$$

which gives

$$p = 2.3026 \operatorname{Log} \frac{A(m+1)}{A(m)} \tag{62}$$

If the variation in $A(m)$ is more rapid we may compute the differential coefficient from the differences according to known methods (cf. f.i., my *Mechanik des Himmels*, Bd. II). For our present purpose the formula (62) is sufficient.

88. *Numerical application.*—The number of stars brighter than the thirteenth, fourteenth, and fifteenth magnitude (photographically) around the Pole is, approximately per square degree, 33.1, 60.4, 110.0. Hence we get the following small table:

m'	13	14	15
$A(m')$	33.1	60.4	110.0
$\operatorname{Log} A(m')$	1.520	1.781	2.041
$\operatorname{Log} A(m'+1) : A(m')$	0.261	0.260	
p	0.600	0.598	

Taking the mean of these values we may suppose

$$p(14) = 0.599$$

Regarding the brighter stars it is more difficult, at least at the pole, to get a satisfactory determination of p, because the number of stars per square degree of this brightness is very small and strongly influenced by accidental errors. From a more comprehensive operation that cannot be outlined here I have obtained, provisionally, at the galactic pole

$$p(5) = 1.336$$

Using now the formula (60) we get

$$\alpha^2 = 10.96$$
$$\alpha = 3.31$$

From the first formula (59) we get

$$m_0 = 5 + 10.96 \times 1.336 - \frac{1}{1.336} = +18.89$$

The second formula (59) leads to the same value of m_0.
We now have the following small table for the computation of N:

m'	13	14	15
$X = \frac{m' - m_0}{\alpha}$	−1.78	−1.48	−1.18
$Q(X)$	0.0375	0.0694	0.1190
$A(m')$	33.1	60.4	110.0
N	883	870	924

so that the mean value of N is

$$N = 892 \text{ per square degree.}$$

There is reason to believe that this number is somewhat too small.

89. *Approximate numerical results.*—The star gauges of the Franklin-Adams charts are at present not complete, therefore the results given below, as well as those already obtained, must be considered as only provisory. They may, however, be used as the basis of some general remarks.

For $p(5)$ and $p(14)$ we get the following provisional values:

North Pole to $b = +50°$; $p(5) = +1.38$; $p(14) = +0.62$
$b = +50°$ to $b = +20°$; $p(5) = +1.17$; $p(14) = +0.72$
Milky Way ; $p(5) = +1.14$; $p(14) = +0.78$

Using these values for $p(5)$ and $p(14)$ we get, according to the formulae in §86, the following values of N, m_0, and α:

b	N	m_0	α
90°−70°	1400	19.7	3.3
70°−60°	1500	19.7	3.3
60°−50°	1800	19.7	3.3
50°−40°	49800	26.6	4.4
40°−30°	64600	26.6	4.4
30°−20°	93300	26.6	4.4
20°−10°	1860000	31.8	4.9
10°− 0°	2650000	31.8	4.9

In this table N indicates the number of stars per square degree.
I repeat that these results are not to be considered as final.

CHAPTER VIII

THE DISTRIBUTION OF THE STARS

90. Let us suppose that the density function, referred to the center of the Galaxy, has the following form:

$$D = D_0 e^{-\frac{1}{2}f} \tag{63}$$

where

$$f = A x^2 + B y^2 + C z^2 + 2 D y z + 2 E z x + 2 F x y \tag{63*}$$

Here x, y, z are the rectangular galactic coördinates of any point in the Galaxy. D_0 is the density in the center of the Galaxy ($x = y = z = 0$).

In this system of coördinates the sun may have the coördinates x_0, y_0, z_0.

From this point (the sun) we assume that we know, by means of the methods given in the preceding paragraph, the number of stars at different distances within a cone having the apex in the sun and an aperture of one square degree.

The density distribution within this cone may be given by the law

$$D = D_1 e^{-\frac{r^2}{2\rho^2}} \tag{64}$$

We suppose further that the value of ρ is determined for a number of cones having their axes directed into different directions in the sky. How is it possible to determine in this way the values of A, B, C, D, E, F, characterizing the general form of the density function (63)?

Let the axis of the cone have the direction cosines

$$\alpha,\ \beta,\ \gamma$$

then the coördinates of any point (x, y, z) in the galactic system are given by the formulae:

$$\begin{aligned} x &= x_0 + r\alpha \\ y &= y_0 + r\beta \\ z &= z_0 + r\gamma \end{aligned} \tag{65}$$

where, as before, r denotes the distance from the sun to the point (x, y, z).

Substituting the values (65) in (63*) we get, after some reductions:

$$f=\frac{r^2}{\rho_1^2}+2r(X_\circ\alpha+Y_\circ\beta+Z_\circ\gamma)+X_\circ x_\circ+Y_\circ y_\circ+Z_\circ z_\circ$$

where

$$\frac{1}{\rho_1^2}=A\alpha^2+B\beta^2+C\gamma^2+2D\beta\gamma+2E\gamma\alpha+2F\alpha\beta \tag{66}$$

$$X_\circ=Ax_\circ+Fy_\circ+Ez_\circ=\frac{1}{2}\frac{\partial f_\circ}{\partial x_\circ}$$

$$Y_\circ=Fx_\circ+By_\circ+Dz_\circ=\frac{1}{2}\frac{\partial f_\circ}{\partial y_\circ} \tag{66*}$$

$$Z_\circ=Ex_\circ+Dy_\circ+Cz_\circ=\frac{1}{2}\frac{\partial f_\circ}{\partial z_\circ}$$

and

$$f_\circ=Ax_\circ^2+By_\circ^2+Cz_\circ^2+2Dy_\circ z_\circ+2Ez_\circ x_\circ+2Fx_\circ y_\circ \tag{66**}$$

91. We have now

$$D=D_\circ e^{-\frac{1}{2}\left[\frac{r^2}{\rho_1^2}+2r(X_\circ\alpha+Y_\circ\beta+Z_\circ\gamma)+X_\circ x_\circ+Y_\circ y_\circ+Z_\circ z_\circ\right]}$$

The coördinates $x_\circ$, $y_\circ$, $z_\circ$ are, however, probably very small. Suppose the center of the Galaxy to coincide with the center of the B stars. We found in §68 this center to be situated at a distance of approximately 18 Siriometers from the sun, a small quantity compared with the dimensions of the Galaxy.

The quantities $X_\circ$, $Y_\circ$, $Z_\circ$, are of the same order of magnitude as $x_\circ$, $y_\circ$, $z_\circ$. Neglecting small quantities of the second order we hence have

$$D=D_\circ e^{-\frac{r^2}{2\rho_1^2}}\left[1-r(X_\circ\alpha+Y_\circ\beta+Z_\circ\gamma)\right] \tag{67}$$

Substituting this value of D in the formula (34) we now get

$$N=\omega D_\circ\left[\sqrt{\frac{\pi}{2}}\,\rho_1^3-2\rho_1^4(X_\circ\alpha+Y_\circ\beta+Z_\circ\gamma)\right] \tag{68}$$

Let us now consider two *opposite* squares in the sky. Let N_1 and N_2 be the total number of stars in these squares, so that

$$N_1=\omega D_\circ\left[\sqrt{\frac{\pi}{2}}\,\rho_1^3-2\rho_1^4(X_\circ\alpha+Y_\circ\beta+Z_\circ\gamma)\right]$$

$$N_2=\omega D_\circ\left[\sqrt{\frac{\pi}{2}}\,\rho_1^3+2\rho_1^4(X_\circ\alpha+Y_\circ\beta+Z_\circ\gamma)\right] \tag{69}$$

Then we have

$$N_1+N_2=\omega D_\circ\sqrt{2\pi}\rho_1^3 \tag{70}$$

$$-N_1+N_2=4\omega D_\circ\rho_1^4(X_\circ\alpha+Y_\circ\beta+Z_\circ\gamma). \tag{70*}$$

The first of these equations gives us the value of ρ_1 (if $D_\circ$ is supposed to be known). Applying the second equation to star counts at different places in the sky, we are able to determine $X_\circ$, $Y_\circ$, $Z_\circ$, and hence from (66*) also $x_\circ$, $y_\circ$, $z_\circ$, or the coördinates of the center of the Galaxy.

The equation (66) is simply the polar equation of the density ellipsoid. Having the value of ρ_1 in different directions, i.e., for a series of values of α, β, γ, we may, according to the method of least squares, determine the quantities

$$A,\ B,\ C,\ D,\ E,\ F$$

which determine the general form of the density ellipsoid.

We may observe that the value of ρ_1 is approximately identical to the quantity ρ used in the preceding chapter. A small correction is to be made because the sun is not situated in the center of the Galaxy, a correction easily obtained from the form (67) for D. In this manner we get a somewhat different form of the equations (38) in §76, which I do not deduce here.

92. There are four parameters to be determined in order to define a sphaeroidal distribution of the stars in the Galaxy, namely:

(1) The mean value ($M_\circ$) of the absolute magnitudes of the stars.

(2) The dispersion (σ) in this magnitude.

(3) The dispersion ρ (or ρ_1) in the density distribution in a cone.

(4) The density ($D_\circ$) at the center of the Galaxy (or in the neighborhood of the sun).

We found in the preceding chapter that the star gauges of the number of stars of different apparent magnitudes give us three relations for the determination of these four parameters. A fourth relation is needed to make the determination complete. Such a relation may be obtained from the mean parallaxes of the stars determined from their proper motions. But it is more satisfactory to determine directly the star density ($D_\circ$) in the neighborhood of the sun, a problem which, with the help of the spectroscopic parallaxes, is now possible to solve approximately. I cannot, however, enter upon this discussion here.

93. *Insufficiency of a sphaeroidal distribution of the stars.*—Although a conclusive discussion of the distribution of the stars in the Galaxy is not possible until the star gauges of the whole sky have been made available, it is possible, however, to draw some plausible conclusions from the preliminary results given in the preceding chapter.

Between the mean apparent magnitude ($m_\circ$) of the stars and their mean absolute magnitude ($M_\circ$) we have, according to §77, the relation:

$$m_\circ = M_\circ + 5 \operatorname{Log} \rho + 0.7922$$

In §89 we have given provisional values of m_0 for different galactic latitudes. For the galactic Pole we thus get the relation

$$19.7 = M_0 + 5 \text{ Log } \rho + 0.7922$$

and for the Milky Way

$$31.8 = M_0 + 5 \text{ Log } \rho + 0.7922$$

The value of m_0 increases 12.1 magnitudes from the galactic Pole to the Milky Way. Hence

$$M_0 + 5 \text{ Log } \rho$$

must also increase by the same amount.

Either M_0 or Log ρ or both must increase. An increase of M_0 is scarcely probable. There are theoretical reasons for believing that the biggest and hottest stars are relatively denser in the Milky Way than at the Poles. If the increase is in ρ alone we ought to have

$$5 \text{ Log } \rho_{MW} - 5 \text{ Log } \rho_{\text{Pole}} = +12.1$$

so that

$$\rho_{MW} = 270 \, \rho_{\text{Pole}}$$

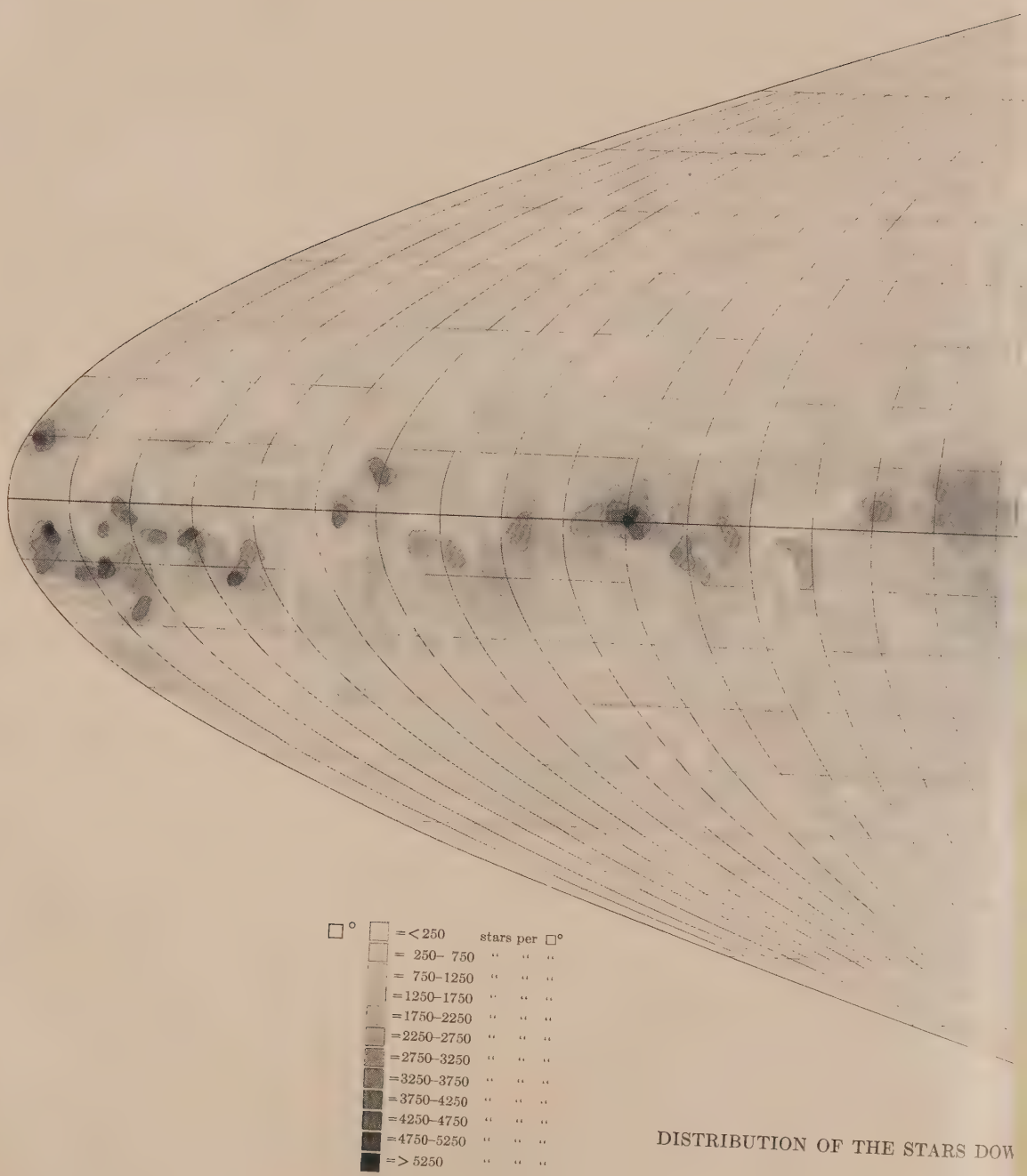
where ρ_{MW} denotes the value of ρ in the Milky Way.

It follows from this relation that the extension of the Galaxy would be more than 270 times greater in the direction of the Milky Way than in the direction of the Poles.

Such a flattening of a sphaeroidal distribution is highly improbable. The existence of the apparent Milky Way with a breadth of twenty degrees or more is directly against such a conclusion.

The great increase in the value of m_0 is most easily explained by giving up the ellipsoidal distribution of the stars in the Galaxy as a whole and exchanging this hypothesis for an ellipsoidal distribution at the center of the Galaxy, which is surrounded by a ring of stars, or better by a ring consisting of clouds of stars. There will be no difficulty in determining the distance of this ring or clouds if the star gauges are extended to stars sufficiently faint.

It is not my purpose to give here the necessary formulae for this determination. But from the plate at the end of this memoir, showing the number of stars down to the fourteenth magnitude (visually) (the plate is based on the star gauges in Lund from the Franklin-Adams charts and reduced to a constant magnitude (14.0) with the help of the star gauges of William and John Herschel), the course of this outer ring may be suggested and also the position and distribution of the different star clouds into which this ring is probably broken up.



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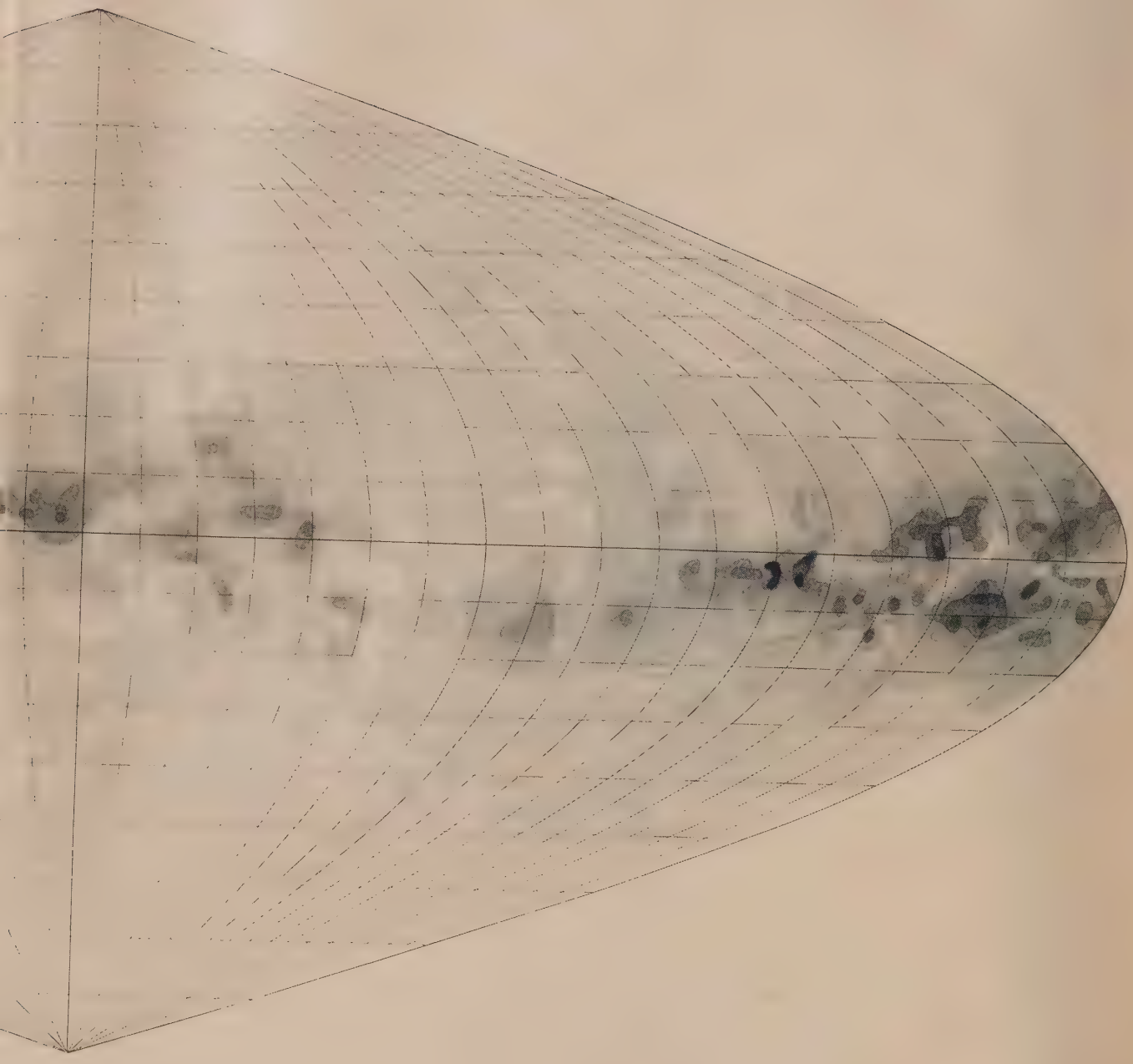

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